PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Bakerian Lecture: On Tidal Prediction

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V. Bakerian Lecture.—*On Tidal Prediction.*

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Introduction.

At most places on the North Atlantic the prediction of high and low water is fairly easy, because there is hardly any diurnal tide. This abnormality makes it sufficient to have a table of the mean fortnightly inequality in the height and interval after lunar transit, supplemented by tables of corrections for the declinations and parallaxes of the disturbing bodies. But when there is a large diurnal inequality, as is commonly the case in other seas, the heights and intervals after the upper and lower lunar transits are widely different; the two halves of each lunation differ much in their characters, and the season of the year has great influence. Thus simple tables, such as are applicable in the absence of diurnal tide, are of no avail.

The tidal information supplied by the Admiralty for such places, consists of rough means of the rise and interval at spring and neap, modified by the important warning that the tide is affected by diurnal inequality. Information of this kind affords scarcely any indication of the time and height of high and low water on any given day, and must, I should think, be almost useless.

This is the present state of affairs at many ports of some importance, but at others a specially constructed tide-table for each day of each year is published in advance. A special tide-table is clearly the best sort of information for the sailor, but the heavy expense of prediction and publication is rarely incurred, except at ports of first rate commercial importance.

There is not, to my knowledge, any arithmetical method in use of computing a special tide-table, which does not involve much work and expense. The admirable tide-predicting instrument of the Indian Government renders the prediction comparatively cheap, yet the instrument can hardly be deemed available for the whole world, and the cost of publication is so considerable, that the instrument cannot, or at least will not, be used for many minor ports at remote places. It is not impossible, too, that national pride may deter the naval authorities of other nations from sending to London for their predictions.*

The object then, of the present paper, is to show how a general tide-table, applicable for all time, may be given in such a form that any one, with an elementary knowledge of the 'Nautical Almanac,' may, in a few minutes, compute two or three tides for the days on which they are required. The tables will also be such that a special tide-table for any year may be computed with comparatively little trouble.

Any tide-table necessarily depends on the tidal constants of the particular port for which it is designed, and it is here supposed that the constants are given in the harmonic system, and are derived from the reduction of tidal observations. Where the observation has been by tide-gauge, the process of reduction is that explained in the Report to the British Association for 1883, but where the observations were only taken at high and low water a different process becomes necessary. I have given, in a previous paper, a scheme of reduction in these cases.†

At ports not of first rate commercial importance, tidal observation has rarely been by tide-gauge, and thus it is exactly at those ports where the method of this paper may prove most useful, that we are deprived of the ordinary methods of harmonic analysis. On this account I regard my previous paper as preliminary to the present one, although the two are logically independent of one another.

In the harmonic method the complete analytical expression for the height of water at any time consists of the sum of a number of heights, each multiplied by the cosine of an angle. Each of these angles or arguments involves some or all of the mean longitudes of moon, sun, lunar perigee, and solar perigee; there are, also, certain corrections depending on the longitude of the moon's nodes. The variability of height of water depends principally on the mean longitudes of moon and sun, and, to a subordinate degree, on the longitudes of lunar perigee and node—for the solar perigee is sensibly fixed. There are, therefore, two principal variables and two subordinate ones besides the time. This statement suggests the construction of a table of double entry for the variability of water due to the principal variables, and of correctional tables for the subordinate ones; and this is the plan developed below.

It would not be appropriate to retain the mean longitudes of moon and sun as principal variables, nor the mean longitude of lunar perigee as one of the subordinate variables. In the final table the principal variables are the time of moon's transit, and the time of year, or strictly speaking the sun's true longitude. As to the sub-

* The instrument may be used, I believe, on the payment of certain fees.

† "On the Harmonic Analysis of Tidal Observations of High and Low Water," 'Roy. Soc. Proc.,' vol. 48, 1890, p. 278.

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ordinate variables, the moon's parallax is taken as one, whilst the longitude of the node is retained as the other.

Throughout the first part of the analytical development the moon's true longitude, measured in the orbit, is taken as one of the principal variables. Transition is then made to the time of transit of a fictitious satellite whose right ascension is equal to the moon's true longitude, and the final transition is to the transits of the real moon. This transposition of variables has necessitated a complete development of the tide-generating forces from the beginning, and thus the analysis of the paper is almost complete in itself, although reference is necessarily made to the harmonic method.

Such a piece of work as the present can only be deemed complete when an example has been worked out to test the accuracy of the tidal prediction, and when rules have been drawn up for the arithmetical processes, forming a complete code of instructions to the computer. The example below is intended to carry out these requirements.

I chose the port of Aden for the example, because its tides are more complex and apparently irregular than those of any other place, which, as far as I know, has been thoroughly treated. The tidal constants for Aden are well determined, and the annual tide-tables of the Indian Government afford the means of comparison between my predictions and those of the tide-predicting instrument.

The arithmetic of the example has been long, and the plan of marshalling the work has been rearranged many times. An ordinary computer is said to work best when he is ignorant of the meaning of his work, but in this kind of tentative work a satisfactory arrangement cannot be attained without a full comprehension of the reason of the method. I was, therefore, very fortunate in securing the enthusiastic assistance of Mr. J. W. F. ALLNUTT, and I owe him my warm thanks for the laborious computations he has carried out. After computing fully half of the original table, he made a comparison for the whole of 1889 between our predictions and those of the Indian Government.

I had hoped that tables, less elaborate than those exhibited below, might have sufficed. It appeared, however, that the changes during the lunation and with the time of year, which affect the height and interval, are so abrupt and so great, that short tables would give very inaccurate results, unless used with elaborate interpolation. We can clearly never expect sailors to use tables of double entry, in which interpolation to second and third differences is required; and indeed any interpolation is objectionable. It is proposed, therefore, that the tables shall be made so full that interpolation will be unnecessary. This plan, of course, throws the whole of the interpolation on to the computer, and although extensive interpolation—even when done graphically—is tedious, yet it is obviously best to have it done once for all, rather than piecemeal by the user of the tables.

The first part of the paper gives the analytical development, the second contains the numerical example and rules of computation, and in the third I give some account of the comparison with other predictions and actuality, and suggestions for abridgements in cases where less accuracy shall be thought sufficient.

The analytical formulæ of the first part are much scattered, and it may not always be easy to see whither they tend; but the second part virtually contains a summary of the first, and a comparison between these two parts will show both the reasons for the analysis and those for the arithmetical processes.

I have tried to make the instructions to the computer so complete that he need not be troubled with the reason for his operations; but if there is, through inadvertence, any incompleteness, reference must be made to the analytical development for interpretation and instruction.

Part I.—Analysis.

§ 1. *Development of the Tide-Generating Potential.*

Let A, B, C (fig. 1) be axes fixed in the earth, AB being the equator and C the north pole.

Let r, ρ be respectively the radius-vectors of M the moon, and of P any other point. Let M_1, M_2, M_3 be the direction cosines of the moon, and ζ_1, ζ_2, ζ_3 those of P, both referred to the axes ABC.

Then by the usual theory the potential V, of the second order of harmonics, is

$$V = \tfrac{3}{2}\frac{\mu M}{r^3}\rho^2(\cos^2 \text{PM} - \tfrac{1}{3}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

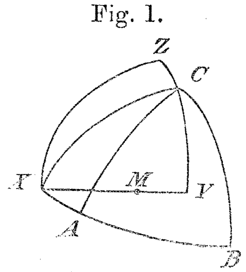
where M is the moon's mass, and μ the attractional constant.

But since

$$\cos \text{PM} = \zeta_1 M_1 + \zeta_2 M_2 + \zeta_3 M_3,$$

$$\cos^2 \text{PM} - \tfrac{1}{3} = 2\zeta_1\zeta_2 . M_1M_2 + 2 \cdot \frac{\zeta_1^2 - \zeta_2^2}{2} \cdot \frac{M_1^2 - M_2^2}{2} + 2\zeta_2\zeta_3 . M_2M_3 + 2\zeta_1\zeta_3 . M_1M_3$$
$$+ \tfrac{3}{2}(\tfrac{1}{3} - \zeta_3^2)(\tfrac{1}{3} - M_3^2) \quad . \quad . \quad . \quad . \quad (2).$$

Fig. 1.



Now let X, Y, Z, fig. 1, be a second set of rectangular axes, fixed in space; let M be the projection of the moon in her orbit XY. Let $I =$ ZC, the obliquity of the lunar orbit to the equator; let $\chi = AX = BCY$, and $\psi = MX$, the moon's longitude measured in her orbit from the point X.

From the figure it is clear that

$$\left.\begin{aligned} M_1 &= \cos\psi\cos\chi + \sin\psi\sin\chi\cos I \\ &= \cos^2\tfrac{1}{2}I\cos(\chi-\psi) + \sin^2\tfrac{1}{2}I\cos(\chi+\psi), \\ M_2 &= -\cos\psi\sin\chi + \sin\psi\cos\chi\cos I \\ &= -\cos^2\tfrac{1}{2}I\sin(\chi-\psi) - \sin^2\tfrac{1}{2}I\sin(\chi+\psi), \\ M_3 &= \sin I\sin\psi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3).$$

Hence from (3)

$$\left.\begin{aligned} M_1^2 - M_2^2 &= \cos^4\tfrac{1}{2}I\cos 2(\chi-\psi) \\ &\qquad + \tfrac{1}{2}\sin^2 I\cos 2\chi + \sin^4\tfrac{1}{2}I\cos 2(\chi+\psi), \\ -2M_1M_2 &= \cos^4\tfrac{1}{2}I\sin 2(\chi-\psi) \\ &\qquad + \tfrac{1}{2}\sin^2 I\sin 2\chi + \sin^4\tfrac{1}{2}I\sin 2(\chi+\psi), \\ M_2M_3 &= -\sin\tfrac{1}{2}I\cos^3\tfrac{1}{2}I\cos(\chi-2\psi) \\ &\qquad + \tfrac{1}{2}\sin I\cos I\cos\chi + \sin^3\tfrac{1}{2}I\cos\tfrac{1}{2}I\cos(\chi+2\psi), \\ M_1M_3 &= -\sin\tfrac{1}{2}I\cos^3\tfrac{1}{2}I\cos(\chi-2\psi) \\ &\qquad + \tfrac{1}{2}\sin I\cos I\sin\chi + \sin^3\tfrac{1}{2}I\cos\tfrac{1}{2}I\sin(\chi+2\psi), \\ \tfrac{1}{3} - M_3^2 &= \tfrac{1}{3} - \tfrac{1}{2}\sin^2 I + \tfrac{1}{2}\sin^2 I\cos 2\psi, \end{aligned}\right\} \quad . \quad (4).$$

In order to simplify the result of the substitutions from (4) into (2) and (1), the axes fixed in the earth may be taken as follows :—

The axis A on the equator in the meridian of the place where the tides are observed, B 90° east of A, and C as already stated at the north pole.

Then if λ be the latitude of the place of observation

$$\zeta_1 = \cos\lambda, \qquad \zeta_2 = 0, \qquad \zeta_3 = \sin\lambda.$$

Hence (2) becomes

$$\cos^2 PM - \tfrac{1}{3} = \tfrac{1}{2}(M_1^2 - M_2^2)\cos^2\lambda + M_1M_3\sin 2\lambda + \tfrac{3}{2}(\tfrac{1}{3} - M_3^2)(\tfrac{1}{3} - \sin^2\lambda) \quad . \quad (5).$$

We shall therefore only require the first and the last two of (4), and these may be considerably simplified.

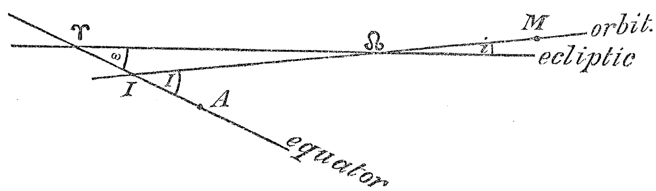
The angle I ranges between 23° 27′ ± 5° 9′; hence the term in $M_1^2 - M_2^2$ which involves $\sin^4\frac{1}{2}I$ is negligible, and similarly that which involves $\sin^3\frac{1}{2}I\cos\frac{1}{2}I$ in M_1M_3 may be omitted.

Further, it is only necessary to develop V in as far as it is variable; now as I ranges from 18° 18′ to 28° 36′, and back again in the course of 19 years, $\sin^2 I$ oscillates slightly in value about a mean. The "tides of long period" corresponding to the term $\frac{1}{3}\sin^2 I \cos 2\psi$ in $\frac{1}{3} - M_3^2$, are, although sensible, so small that we shall take no account of them; *à fortiori* the variability of $\frac{1}{3} - \frac{1}{2}\sin^2 I$ is negligible. It would be easy, however, to take account of these terms if it were desirable to do so.

We need therefore only pay attention to the first and fourth of (4), and the last term in each may be omitted.

In fig. 2 let M be the moon in her orbit, and let A be the axis A of fig. 1 fixed in the earth.

Fig. 2.



Then let ν be the right ascension of the point I, and ξ its longitude measured in the moon's orbit.

Let ω be the obliquity of the ecliptic, i the inclination of the moon's orbit, ☊ the longitude of the moon's node.

Let t be the mean solar hour angle at the place and time of observation, h the sun's mean longitude, and l the moon's longitude measured in the orbit.

Then, by the definition of χ and ψ, we have

$$\chi = \mathrm{IA}, \qquad \psi = \mathrm{IM}.$$

Since $\nu = ♈\mathrm{I}$, $\xi = ♈☊ - ☊\mathrm{I}$, $♈☊ = ☊$, and $t + h$ the sidereal hour angle,

$$\chi = t + h - \nu, \qquad \psi = l - \xi.$$

Thus to the degree of approximation adopted, (4) becomes

$$\left.\begin{aligned} M_1^2 - M_2^2 &= \cos^4 \tfrac{1}{2} I \cos 2(t + h - l - \nu + \xi) + \tfrac{1}{2}\sin^2 I \cos 2(t + h - \nu), \\ M_1 M_3 &= \cos^3 \tfrac{1}{2} I \sin \tfrac{1}{2} I \cos (t + h - 2l + \tfrac{1}{2}\pi - \nu + 2\xi) \\ &\quad + \tfrac{1}{2}\sin I \cos I \cos (t + h - \tfrac{1}{2}\pi - \nu), \\ \tfrac{1}{3} - M_3^2 &= 0, \end{aligned}\right\} \quad (6).$$

We must now obtain approximate formulæ to express the functions of I, ν, and ξ in terms of ω, i, and ☊. The inclination i may be treated as so small that its square may be neglected.

From fig. 2 we have

$$\cos I = \cos i \cos \omega - \sin i \sin \omega \cos ☊,$$

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whence

$$\left.\begin{aligned}
\cos I &= \cos\omega - i\sin\omega\cos ☊\,,\\
\cos^2\tfrac{1}{2}I &= \cos^2\tfrac{1}{2}\omega\,(1 - i\tan\tfrac{1}{2}\omega\cos ☊\,),\\
\sin^2\tfrac{1}{2}I &= \sin^2\tfrac{1}{2}\omega\,(1 + i\cot\tfrac{1}{2}\omega\cos ☊\,),\\
\cos^4\tfrac{1}{2}I &= \cos^4\tfrac{1}{2}\omega\,(1 - 2i\tan\tfrac{1}{2}\omega\cos ☊\,),\\
\sin^2 I &= \sin^2\omega\,(1 + 2i\cot\omega\cos ☊\,),\\
\cos^3\tfrac{1}{2}I\sin\tfrac{1}{2}I &= \cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\left(1 + i\frac{2\cos\omega - 1}{\sin\omega}\cos ☊\right),\\
\sin I\cos I &= \sin\omega\cos\omega\,(1 + 2i\cot 2\omega\cos ☊\,),
\end{aligned}\right\} \quad . \quad . \quad (7).$$

Again from the figure

$$\frac{\sin\nu}{\sin i} = \frac{\sin ☊}{\sin I} = \frac{\sin(☊ - \xi)}{\sin\omega}\,.$$

Since ξ is small,

$$1 - \xi\cot ☊ = \frac{\sin\omega}{\sin I} = 1 - i\cot\omega\cos ☊\,,$$

$$\nu = \frac{\sin i\sin ☊}{\sin I}\,.$$

Therefore

$$\left.\begin{aligned}
\nu &= i\,\frac{\sin ☊}{\sin\omega},\\
\xi &= i\cot\omega\sin ☊\,,\\
\nu - \xi &= i\tan\tfrac{1}{2}\omega\sin ☊\,,\\
2\xi - \nu &= i\,\frac{2\cos\omega - 1}{\sin\omega}\sin ☊\,,
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Hence from (7) and (8)

$$\left.\begin{aligned}
\cos^4\tfrac{1}{2}I\cos 2(\nu - \xi) &= \cos^4\tfrac{1}{2}\omega\,(1 - 2i\tan\tfrac{1}{2}\omega\cos ☊\,),\\
\cos^4\tfrac{1}{2}I\sin 2(\nu - \xi) &= \cos^4\tfrac{1}{2}\omega\,.\,2i\tan\tfrac{1}{2}\omega\sin ☊\,,\\
\tfrac{1}{2}\sin^2 I\cos 2\nu &= \tfrac{1}{2}\sin^2\omega\,(1 + 2i\cot\omega\cos ☊\,),\\
\tfrac{1}{2}\sin^2 I\sin 2\nu &= \tfrac{1}{2}\sin^2\omega\,.\,2i\,\mathrm{cosec}\,\omega\sin ☊\,,\\
\cos^3\tfrac{1}{2}I\sin\tfrac{1}{2}I\cos(2\xi - \nu) &= \cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\,(1 + i(2\cos\omega - 1)\,\mathrm{cosec}\,\omega\cos ☊\,),\\
\cos^3\tfrac{1}{2}I\sin\tfrac{1}{2}I\sin(2\xi - \nu) &= \cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\,.\,i\,(2\cos\omega - 1)\,\mathrm{cosec}\,\omega\sin ☊\,,\\
\tfrac{1}{2}\sin I\cos I\cos\nu &= \tfrac{1}{2}\sin\omega\cos\omega\,(1 + 2i\cot 2\omega\cos ☊\,),\\
\tfrac{1}{2}\sin I\cos I\sin\nu &= \tfrac{1}{2}\sin\omega\cos\omega\,.\,i\,\mathrm{cosec}\,\omega\sin ☊\,,
\end{aligned}\right\} (9).$$

Now $\omega = 23°\ 27'{\cdot}3$, $\quad i = 5°\ 8'{\cdot}8$; whence

$$2i\tan\tfrac{1}{2}\omega = {\cdot}0372,\ 2i\cot\omega = \frac{{\cdot}283}{{\cdot}683},\ 2i\,\mathrm{cosec}\,\omega = \frac{{\cdot}308}{{\cdot}683},\ i\,\frac{2\cos\omega - 1}{\sin\omega} = {\cdot}188,$$

$$2i\cot 2\omega = \frac{{\cdot}115}{{\cdot}683},\ i\,\mathrm{cosec}\,\omega = \frac{{\cdot}154}{{\cdot}683}\,.$$

Four of these are written with ·683 in the denominator for reasons given below. Then from (9) and (6) we have

$$\left.\begin{aligned}
M_1^2 - M_2^2 = {} & \cos^4 \tfrac{1}{2}\omega \cos 2(t+h-l) + \tfrac{1}{2}\sin^2\omega \cos 2(t+h) \\
& + \left\{ -\cos^4\tfrac{1}{2}\omega \,.\, {\cdot}0372 \cos 2(t+h-l) \right. \\
& \qquad \left. + \tfrac{1}{2}\sin^2\omega \,.\, \frac{{\cdot}283}{{\cdot}683}\cos 2(t+h)\right\}\cos\Omega \\
& + \left\{ \cos^4\tfrac{1}{2}\omega \,.\, {\cdot}0372 \sin 2(t+h-l) \right. \\
& \qquad \left. + \tfrac{1}{2}\sin^2\omega \,.\, \frac{{\cdot}308}{{\cdot}683}\sin 2(t+h)\right\}\sin\Omega \,. \\
M_1 M_3 = {} & \cos^3\tfrac{1}{2}\omega \sin\tfrac{1}{2}\omega \cos(t+h-2l+\tfrac{1}{2}\pi) \\
& \qquad + \tfrac{1}{2}\sin\omega\cos\omega\cos(t+h-\tfrac{1}{2}\pi) \\
& + \left\{ \cos^3\tfrac{1}{2}\omega \sin\tfrac{1}{2}\omega \,.\, {\cdot}188\cos(t+h-2l+\tfrac{1}{2}\pi) \right. \\
& \qquad \left. + \tfrac{1}{2}\sin\omega\cos\omega \,.\, \frac{{\cdot}115}{{\cdot}683}\cos(t+h-\tfrac{1}{2}\pi)\right\}\cos\Omega \\
& + \left\{ -\cos^3\tfrac{1}{2}\omega \sin\tfrac{1}{2}\omega \,.\, {\cdot}188\sin(t+h-2l+\tfrac{1}{2}\pi) \right. \\
& \qquad \left. + \tfrac{1}{2}\sin\omega\cos\omega \,.\, \frac{{\cdot}154}{{\cdot}683}\sin(t+h-\tfrac{1}{2}\pi)\right\}\sin\Omega
\end{aligned}\right\} \quad (10).$$

Now let c be the moon's mean distance, and let $\tau = \frac{3}{2}\mu M/c^3$, and

$$1 + P = \frac{c^3}{r^3}\,.$$

Then as far as concerns the moon

$$V = \tau\rho^2(1+P)\{\tfrac{1}{2}\cos^2\lambda\,(M_1^2 - M_2^2) + \sin 2\lambda \,.\, M_1 M_3\} \quad . \quad . \quad . \quad (11),$$

where $M_1^2 - M_2^2$, M_1M_3 are given by (10).

In writing down the corresponding functions for the sun, we shall write a subscript accent to all the symbols, and accordingly V for the sun is given by

$$V_{,} = \tau_{,}\rho^2(1+P_{,})\{\tfrac{1}{2}\cos^2\lambda\,(M_{,1}^2 - M_{,2}^2) + \sin 2\lambda \,.\, M_{,1} M_{,3}\} \quad . \quad . \quad (12),$$

where, by symmetry with (10)

$$\left.\begin{aligned}
M_{,1}^2 - M_{,2}^2 &= \cos^4\tfrac{1}{2}\omega\cos 2(t+h-l_{,}) + \tfrac{1}{2}\sin^2\omega\cos 2(t+h), \\
M_{,1}M_{,3} &= \cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega\cos(t+h-2l_{,}+\tfrac{1}{2}\pi) \\
&\qquad + \tfrac{1}{2}\sin\omega\cos\omega\cos(t+h-\tfrac{1}{2}\pi),
\end{aligned}\right\} \quad . \quad (13).$$

The masses and mean distances of the moon and sun are such that

$$\frac{\tau}{\tau+\tau_{,}} = \cdot 683, \quad \frac{\tau_{,}}{\tau+\tau_{,}} = \cdot 317 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Strictly speaking, τ denotes $\frac{3}{2}\mu M c^{-3}(1+\frac{3}{2}e^2)$, and $\tau_{,}$ denotes $\frac{3}{2}\mu M_{,} c_{,}^{-3}(1+\frac{3}{2}e_{,}^2)$; where e, $e_{,}$ are the eccentricities of the lunar and solar orbits. A comparison with the Report on Tides to the British Association for 1883, will show that, for the purpose in hand, it is correct to introduce these factors involving e^2 and $e_{,}^2$.

The whole potential may be collected from (10), (11), (12), (13), (14).

Since P is small the factor $(1+P)$ may be treated as unity in the small terms which involve cos ☊ and sin ☊, and we shall find it convenient to distinguish the several parts of V, denoting by V the principal part, by V_P the part involving P, by $V_{☊}$ the part involving ☊, and by $V_{P_{,}}$ the part involving $P_{,}$.

For brevity write

$$K = \rho^2 . \tfrac{1}{2}\cos^2\lambda . \cos^4\tfrac{1}{2}\omega; \qquad K_{,,} = \rho^2 . \tfrac{1}{2}\cos^2\lambda . \tfrac{1}{2}\sin^2\omega;$$

$$K_o = \rho^2 . \sin 2\lambda . \cos^3\tfrac{1}{2}\omega\sin\tfrac{1}{2}\omega; \qquad K_{,} = \rho^2 . \sin 2\lambda . \tfrac{1}{2}\sin\omega\cos\omega.$$

Then remarking that $\tau(1+P)+\tau_{,}(1+P_{,}) = (\tau+\tau_{,})(1+\cdot 683P)(1+\cdot 317P_{,})$ approximately, we have

$$\begin{aligned} V = {} & \tau K\cos 2(t+h-l) + (\tau+\tau_{,})K_{,,}\cos 2(t+h) + \tau_{,}K\cos 2(t+h-l_{,}) \\ & + \tau K_o\cos(t+h-2l+\tfrac{1}{2}\pi) + (\tau+\tau_{,})K_{,}\cos(t+h-\tfrac{1}{2}\pi) \\ & + \tau_{,}K_o\cos(t+h-2l_{,}+\tfrac{1}{2}\pi). \end{aligned}$$

$$\begin{aligned} V_P = {} & P\{\tau K\cos 2(t+h-l) + \cdot 683(\tau+\tau_{,})K_{,,}\cos 2(t+h) \\ & + \tau K_o\cos(t+h-2l+\tfrac{1}{2}\pi) + \cdot 683(\tau+\tau_{,})K_{,}\cos(t+h-\tfrac{1}{2}\pi)\}. \end{aligned}$$

$$\begin{aligned} V_{☊} = {} & \cos ☊ \{-\tau K \cdot 0372\cos 2(t+h-l) + (\tau+\tau_{,})K_{,,}\cdot 283\cos 2(t+h) \\ & \quad + \tau K_o \cdot 188\cos(t+h-2l+\tfrac{1}{2}\pi) + (\tau+\tau_{,})K_{,}\cdot 115\cos(t+h-\tfrac{1}{2}\pi\} \\ & + \sin ☊ \{+\tau K\cdot 0372\sin 2(t+h-l) + (\tau+\tau_{,})K_{,,}\cdot 308\sin 2(t+h) \\ & \quad - \tau K_o\cdot 188\sin(t+h-2l+\tfrac{1}{2}\pi) + (\tau+\tau_{,})K_{,}\cdot 154\sin(t+h-\tfrac{1}{2}\pi)\}. \end{aligned}$$

$$\begin{aligned} V_{P_{,}} = {} & P_{,}\{\cdot 317(\tau+\tau_{,})K_{,,}\cos 2(t+h) + \tau_{,}K\cos 2(t+h-l_{,}) \\ & + \cdot 317(\tau+\tau_{,})K_{,}\cos(t+h-\tfrac{1}{2}\pi) + \tau_{,}K_o\cos(t+h-2l_{,}+\tfrac{1}{2}\pi)\}. \quad . \quad (15). \end{aligned}$$

The whole tide-generating potential is then $V + V_P + V_{☊} + V_{P_{,}}$.

§ 2. *Formula for the Height of Tide.*

If we had been going to consider the complete expression for the tide in terms of a series of simple harmonic functions of the time, it would have been necessary to substitute for l, $l_{,}$, P, $P_{,}$ their values in terms of the mean longitudes s and h of the two bodies, and of the eccentricities e, $e_{,}$ of their orbits. When the potential is so expanded the principle of forced oscillations allows us to conclude that the oscillations of the sea will be of the same periods and types as the several terms of the potential, but with amplitudes and phases which can only be deduced from observation. The oscillations of the sea will not however be necessarily of the simple harmonic form, and accordingly "over-tides" of double and triple frequency have to be introduced in order to represent the motion according to FOURIER's method.

This is the plan pursued in the "harmonic analysis of tidal observations," and each simple harmonic oscillation is known by an arbitrarily chosen initial letter.

It is found in fact, as is suggested by theory, that tides of approximately the same frequency or "speed" have amplitudes approximately proportional to their corresponding terms in the potential, and have their phases retarded by approximately the same amount.

The notation of harmonic analysis will be adopted here, because it is proposed to compute the tide table from the harmonic constants.

The mean longitudes of the sun, moon, and lunar perigee are denoted by h, s, p, and their hourly changes by σ, η, ϖ (from σελήνη, ἥλιος, perigee); the angular velocity of the earth's rotation is γ (from γῆ), so that $\gamma - \eta$ is 15° per hour.

The eccentricities of the lunar and solar orbits are e, $e_{,}$.

In the harmonic system the tides are denominated by the arbitrarily chosen initials M_2, M_4, S_2, S_4, K_2, K_1, N, L, ν, λ, μ, T, R, O, P, Q, J, Sa, Ssa, and the semi-ranges of these tides will be here denoted by the suffixes m, $2m$, s, $2s$, n, l, ν, λ, μ, t, r, o, p, q, j, sa, ssa to the symbol H, and the retardations of phase by the same suffixes to the symbol κ. But the semi-ranges and retardations of the tides K_2, K_1 are denoted by $H_{,,}$, $\kappa_{,,}$, $H_{,}$, $\kappa_{,}$ instead of conforming to the rest of the notation.*

Then all the H's and κ's are the immediate results of harmonic analysis, and are supposed to be given when the construction of a tide-table is contemplated.

In the development (15) of V, the first term corresponds with the principal lunar tide M_2, its over-tide M_4, parts of the elliptic tides N, L, parts of the evectional tides ν, λ, and part of the variational tide μ.

The first term of V_P contributes the remainder of N, L, ν, λ, μ.

The second term of V corresponds with the luni-solar semi-diurnal tide K_2, and the second term of V_P corresponds with small inequalities in K_2, which are neglected in the harmonic analysis.

* I adopted this originally for convenience in writing, and having got used to the notation shall retain it.

The third term of V corresponds with the principal solar tide S_2, its over-tide S_4, and parts of the elliptic tides T, R.

The fourth term of V corresponds with the lunar diurnal tide O, and parts of the elliptic tides M_1 and Q; the third term of V_P corresponds with another part of M_1 and the rest of Q.

The fifth term of V corresponds with the luni-solar diurnal tide K_1, and with part of the elliptic tides M_1 and J; its over-tide fuses with K_2. The fourth term of V_P corresponds with part of the elliptic tide M_1, and the rest of J.

The last term of V corresponds with the solar diurnal tide P, and with small inequalities neglected in the harmonic analysis.

The whole of $V_{☊}$ corresponds with those factors of augmentation and small alterations of phase, which are denoted by f and functions of ν and ξ in harmonic analysis.

$V_{P_,}$ corresponds with the remainder of the solar elliptic tides T, R and with other small inequalities usually neglected in the harmonic analysis.

The semi-annual (Ssa) and annual (Sa) tides are due to meteorological causes, and have to be introduced after we have done with astronomical considerations, so that they do not enter through the V's.

Since the terms V, V_P, $V_{☊}$, $V_{P_,}$ are approximately simple harmonics, it follows that each term will correspond to an approximately simple harmonic tide, with amplitude and phase the same as that given by harmonic analysis, and to its over-tides. If then we write

$$\Delta = 2t + 2h - 2l - \kappa_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

the height of water corresponding to V is,

$$\begin{aligned} h = {} & \mathrm{H}_m \cos \Delta + \mathrm{H}_{,,} \cos (\Delta + \kappa_m + 2l - \kappa_{,,}) + \mathrm{H}_s \cos (\Delta + \kappa_m + 2l - 2l_, - \kappa_s) \\ & + \mathrm{H}_o \cos [\tfrac{1}{2} (\Delta + \kappa_m) - l + \tfrac{1}{2}\pi - \kappa_o] + \mathrm{H}_, \cos [\tfrac{1}{2} (\Delta + \kappa_m) + l - \tfrac{1}{2}\pi - \kappa_,] \\ & + \mathrm{H}_p \cos [\tfrac{1}{2} (\Delta + \kappa_m) + l - 2l_, + \tfrac{1}{2}\pi - \kappa_p] + \mathrm{H}_{2m} \cos [2(\Delta + \kappa_m) - \kappa_{2m}] \\ & + \mathrm{H}_{2s} \cos [2 (\Delta + \kappa_m) + 4l - 4l_, - \kappa_{2s}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17). \end{aligned}$$

We shall, in the developments immediately following, leave $V_{P_,}$ out of consideration, and shall resume the effects of the sun's parallax in § 11.

In order to represent, as far as possible, the elliptic tides, it is found to be necessary to introduce certain numerical factors α, β, and certain angles ${}_m\kappa$, ${}_o\kappa$, as shown in the next formula. It is also practically convenient that P should denote the value of c^3/r^3, not at the time corresponding to t, but some hours earlier.

Then the height of water corresponding to V_P is taken as

$$\begin{aligned} h_P = P \{ & \alpha \mathrm{H}_m \cos (\Delta + \kappa_m - {}_m\kappa) + \cdot 683\, \mathrm{H}_{,,} \cos (\Delta + \kappa_m + 2l - \kappa_{,,}) \\ & + \beta \mathrm{H}_o \cos [\tfrac{1}{2} (\Delta + \kappa_m) - l + \tfrac{1}{2}\pi - {}_o\kappa] \\ & + \cdot 683\, \mathrm{H}_, \cos [\tfrac{1}{2} (\Delta + \kappa_m) + l - \tfrac{1}{2}\pi - \kappa_,] \} \quad . \quad . \quad . \quad . \quad . \quad (18). \end{aligned}$$

The height corresponding to $V_{☊}$ is

$$h_{☊} = \cos ☊ \{ - H_m \cdot 0372 \cos \Delta + H_{\prime\prime} \cdot 283 \cos (\Delta + \kappa_m + 2l - \kappa_{\prime\prime}) + H_o \cdot 188 \cos [\tfrac{1}{2}(\Delta + \kappa_m) - l + \tfrac{1}{2}\pi - \kappa_o] + H_{\prime} \cdot 115 \cos [\tfrac{1}{2}(\Delta + \kappa_m) + l - \tfrac{1}{2}\pi - \kappa_{\prime}]\} + \sin ☊ \{H_m \cdot 0372 \sin \Delta + H_{\prime\prime} \cdot 308 \sin (\Delta + \kappa_m + 2l - \kappa_{\prime\prime}) - H_o \cdot 188 \sin [\tfrac{1}{2}(\Delta + \kappa_m) - l + \tfrac{1}{2}\pi - \kappa_o] + H_{\prime} \cdot 154 \sin [\tfrac{1}{2}(\Delta + \kappa_m) + l - \tfrac{1}{2}\pi - \kappa_{\prime}]\} \quad . \quad . \quad . \quad (19).$$

§ 3. *The Elliptic Tides.*

The constants α, β, ${}_m\kappa$, ${}_o\kappa$, are insufficient to fully represent the harmonic tides, N, L, ν, λ, μ, Q, J, and it would render the proposed method of computing a tide-table too cumbrous for practical use if additional constants were introduced. It is, therefore, necessary to adopt a compromise.

The tides N and L of the harmonic system are given by

$$H_n \cos [2t + 2h - 2s - (s - p) - \kappa_n] + H_l \cos [2t + 2h - 2s + (s - p) + \pi - \kappa_l].$$

In the present development the mean lunar tide M_2 and the elliptic tides are to be represented by

$$H_m \cos [2t + 2h - 2l - \kappa_m] + \alpha\, H_m P \cos [2t + 2h - 2l - {}_m\kappa].$$

It has been already remarked that it will be convenient that P should refer to a time earlier than t, and the time referred to is that of the moon's transit. Now H. W. occurs later than moon's transit by an interval whose mean value is $\kappa_m/2(\gamma - \sigma)$, and the mean value of the interval to the preceding L. W. is $(\kappa_m - \pi)/2(\gamma - \sigma)$. If, therefore, we put $\zeta = \frac{\sigma - \varpi}{2(\gamma - \sigma)}\kappa_m$ for H. W., and $\frac{\sigma - \varpi}{2(\gamma - \sigma)}(\kappa_m - \pi)$ for L. W., we have approximately

$$P = \frac{c^3}{r^3} - 1 = 3e \cos [s - p - \zeta],$$

also

$$2l = 2s + 4e \sin (s - p).$$

Hence the elliptic tides are represented in the present development by

$$2e_m H \cos (2t + 2h - 3s + p - \kappa_m) + \tfrac{3}{2}\alpha e H_m \cos (2t + 2h - 3s + p - {}_m\kappa + \zeta) + 2eH_m \cos (2t + 2h - s - p + \pi - \kappa_m) - \tfrac{3}{2}\alpha e H_m \cos (2t + 2h - s - p + \pi - {}_m\kappa - \zeta).$$

In order that the first pair of these terms may give the N tide correctly we must have

$$H_n = 2eH_m \cos(\kappa_n - \kappa_m) + \tfrac{3}{2}\alpha e H_m \cos(\kappa_n - {}_m\kappa + \zeta),$$
$$0 = 2eH_m \sin(\kappa_n - \kappa_m) + \tfrac{3}{2}\alpha e H_m \sin(\kappa_n - {}_m\kappa + \zeta).$$

And the condition that the second pair may give the L tide is

$$H_l = 2eH_m \cos(\kappa_l - \kappa_m) - \tfrac{3}{2}\alpha e H_m \cos(\kappa_l - {}_m\kappa - \zeta),$$
$$0 = 2eH_m \sin(\kappa_l - \kappa_m) - \tfrac{3}{2}\alpha e H_m \sin(\kappa_l - {}_m\kappa - \zeta).$$

The condition for N may be written

$$\tan(\kappa_n - {}_m\kappa + \zeta) = \frac{2eH_m \sin(\kappa_m - \kappa_n)}{H_n - 2eH_m \cos(\kappa_m - \kappa_n)},$$
$$\tfrac{3}{2}\alpha e H_m = \{H_n - 2eH_m \cos(\kappa_m - \kappa_n)\} \sec(\kappa_n - {}_m\kappa + \zeta).$$

Then, since both $\kappa_n - \kappa_m$ and $\kappa_n - {}_m\kappa + \zeta$ are small angles, we have approximately

$${}_m\kappa = \kappa_n + \zeta - \frac{2eH_m(\kappa_m - \kappa_n)}{H_n - 2eH_m},$$
$$\alpha H_m = \frac{2}{3e}(H_n - 2eH_m).$$

Similarly the condition for L gives

$${}_m\kappa = \kappa_l - \zeta - \frac{2eH_m(\kappa_l - \kappa_m)}{2eH_m - H_l},$$
$$\alpha H_m = \frac{2}{3e}(2eH_m - H_l).$$

These conditions cannot be satisfied simultaneously and a compromise must be adopted.

If H_l were zero we should have $\alpha = \tfrac{4}{3}$. Hence if α be taken as greater than $\tfrac{4}{3}$ we are virtually making the L tide negative. Now, it is unadvisable to take the L tide as negative even if small, because, if so, we are representing the elliptic tide compounded from N and L as greatest when it should be smallest and *vice versâ*.

I therefore propose to use the condition for the L tide only for the purpose of putting a superior limit to α, and to the constants to be derived from the condition for the more important N tide.

In order to express the result in the form which will ultimately be required I write

$$p_n = \frac{2\gamma - 3\sigma + \varpi}{2\gamma - 2\sigma} \qquad p_m = \frac{2\gamma - 2\sigma}{2\gamma - 2\sigma} = 1,$$

so that p_n is the ratio of the "speed" of the N tide to that of M_2, and p_m is merely introduced for the sake of algebraic symmetry.

Then

$$\zeta = (p_m - p_n)\,\kappa_m \text{ for H. W. and } (p_m - p_n)(\kappa_m - \pi) \text{ for L. W.}$$

We thus have approximately, when referring to H. W.,

$$p_m \kappa_m - {}_m\kappa = p_n \kappa_m - \kappa_n + \frac{2e\mathrm{H}_m(\kappa_m - \kappa_n)}{\mathrm{H}_n - 2e\mathrm{H}_m},$$

and

$$\alpha \mathrm{H}_m \text{ is equal to the smaller of } \frac{2}{3e}(\mathrm{H}_n - 2e\mathrm{H}_m) \text{ and } \tfrac{4}{3}\mathrm{H}_m.$$

When referring to L. W. we have $(\kappa_m - \pi)$ in the above formula in the place of κ_m wherever κ_m occurs multiplied by a p.

A similar argument may be followed with regard to the diurnal tides, save that the smaller tide corresponding to L, has not usually been evaluated in the harmonic analysis. The tide corresponding to N bears the initial Q, and

$$p_q = \frac{\gamma - 3\sigma + \varpi}{2\gamma - 2\sigma}, \quad p_o = \frac{\gamma - 2\sigma}{2\gamma - 2\sigma},$$

so that p_q and p_o are the ratios of the "speeds" of the Q and O tides to that of M_2. Then

$$\frac{\sigma - \varpi}{2(\gamma - \sigma)} = p_o - p_q,$$

and we find for H. W.

$$p_o \kappa_m - {}_o\kappa = p_q \kappa_m - \kappa_q + \frac{2e\mathrm{H}_o(\kappa_o - \kappa_q)}{\mathrm{H}_q - 2e\mathrm{H}_o},$$

and

$$\beta \mathrm{H}_o \text{ is equal to the smaller of } \frac{2}{3e}(\mathrm{H}_q - 2e\mathrm{H}_o) \text{ and } \tfrac{4}{3}\mathrm{H}_o.$$

Since $e = {\cdot}0549$, $\frac{2}{3e} = 12{\cdot}144 = 19 \times {\cdot}639$ (the factor being introduced because we shall hereafter have to divide by 19); also $\frac{4}{3} = 19 \times \frac{4}{57} = 19 \times {\cdot}070$, and $\frac{1}{2e} = 9{\cdot}11$.

If we put then $\alpha' = \frac{1}{19}\alpha$, $\beta' = \frac{1}{19}\beta$, our formulæ for H. W. become

$$\left.\begin{aligned} p_m \kappa_m - {}_m\kappa &= p_n \kappa_m - \kappa_n + \frac{(\kappa_m - \kappa_n)}{9{\cdot}11\,\mathrm{H}_n/\mathrm{H}_m - 1}, \\ p_o \kappa_m - {}_o\kappa &= p_q \kappa_m - \kappa_q + \frac{(\kappa_o - \kappa_q)}{9.11\,\mathrm{H}_q/\mathrm{H}_o - 1}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (20).$$

where $p_m = 1$, $p_n = \cdot 981$, $p_o = \cdot 481$, $p_q = \cdot 462$, and where for L. W. κ_m is to be replaced by $\kappa_m - \pi$ wherever it is multiplied by a p.

Also, if $36\cdot 4\ \mathrm{H}_n$ is less than $8\mathrm{H}_m$,

$$\left.\begin{aligned} \alpha &= 12\cdot 14\,\frac{\mathrm{H}_n}{\mathrm{H}_m} - \tfrac{4}{3} \\ \alpha'\,\mathrm{H}_n &= \cdot 639\,\mathrm{H}_m - \cdot 070\,\mathrm{H}_m \\ \text{If } 36\cdot 4\ \mathrm{H}_q &\text{ is less than } 8\mathrm{H}_o \\ \beta &= 12\cdot 14\,\frac{\mathrm{H}_q}{\mathrm{H}_o} - \tfrac{4}{3} \\ \beta'\,\mathrm{H}_o &= \cdot 639\,\mathrm{H}_q - \cdot 070\,\mathrm{H}_o \end{aligned}\right\} \qquad (21).$$

If $36\cdot 4\ \mathrm{H}_n$ is greater than $8\ \mathrm{H}_m$, $\alpha = \frac{4}{3}$, and $\alpha'\,\mathrm{H}_m = \cdot 070\ \mathrm{H}_m$.

If $36\cdot 4\ \mathrm{H}_q$ is greater than $8\ \mathrm{H}_o$, $\beta = \frac{4}{3}$, and $\beta'\,\mathrm{H}_o = \cdot 070\ \mathrm{H}_o$.

Lastly, if the elliptic tides are unknown, $\alpha = \beta = 1$, $\alpha' = \beta' = \frac{1}{19}$.

§ 4. *Reference to Transit of Fictitious Moon.*

Suppose that there is a fictitious satellite whose R. A. is equal to the Moon's longitude, and let $l^{(o)}$, $h^{(o)}$ be the R. A. of the fictitious moon and of the mean sun at noon of the day under consideration, and let τ be the mean solar time of the fictitious moon's transit.

Then at transit

$$t + h - l = q\pi,$$

where q is even for upper and odd for lower transit.

The rate of change of $t + h$ is γ, and if $\dot{l}$ be the rate of change of l, τ is given by

$$\tau = \frac{l^{(o)} - h^{(o)} + q\pi}{\gamma - \dot{l}}.$$

Then at time $\tau + T$

$$\begin{aligned} \Delta &= 2t + 2h - 2l - \kappa_m \\ &= 2q\pi + 2(\gamma - \dot{l})\,T - \kappa_m, \end{aligned}$$

and

$$\tfrac{1}{2}\Delta = q\pi + (\gamma - \dot{l})\,T - \tfrac{1}{2}\kappa_m.$$

Now Δ occurs in the arguments of all the semi-diurnal tides, $\frac{1}{2}\Delta$ in those of the diurnal, and 2Δ in those of the quater-diurnal.

Hence we may omit the $2q\pi$ in the expression for Δ, and write

$$\Delta = 2(\gamma - \dot{l})\,T - \kappa_m,$$

provided that when q is odd we change the signs of the diurnal terms.

Therefore we may write the diurnal terms with alternative signs $\pm$, and the upper sign will be appropriate when we refer to an upper transit, and the lower sign to a lower transit.

In the application of the formulæ $T + \tau$ will be the time for which the mean solar hour angle is t, and we shall have

$$T = \frac{\kappa_m}{2(\gamma - \dot{l})} + \frac{\Delta}{2(\gamma - \dot{l})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

The first of these terms is obviously the interval from fictitious transit to the mean lunar H. W., and the second is the interval from lunar H. W. to the time t.

When we wish to discuss low waters it is convenient to put Δ equal to $\delta - \pi$, and in this case we shall have

$$T = \frac{\kappa_m - \pi}{2(\gamma - \dot{l})} + \frac{\delta}{2(\gamma - \dot{l})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

The first of these terms is the interval from transit to the mean lunar L. W., which precedes the H. W. given by (22), and the second is the interval after lunar L. W. to the time t.

We shall proceed to consider H. W., and deduce therefrom the formulæ for L. W.

§ 5. *Reduction of Longitudes of Moon and Sun to time of Fictitious Transit.*

We have seen in (22) that the interval, from fictitious moon's transit to the time t, is

$$T = \frac{\Delta + \kappa_m}{2(\gamma - \dot{l})}.$$

Now the equation of conservation of areas for the moon's motion gives

$$r^2\dot{l} = \sigma c^2 \sqrt{(1 - e^2)},$$

but since $c^3/r^3 = 1 + P$, and since P is small, we have

$$\dot{l} = \sigma(1 + \tfrac{2}{3}P), \text{ very nearly.}$$

Therefore

$$T = \frac{\Delta + \kappa_m}{2(\gamma - \sigma)}\left(1 + \frac{2}{3}\frac{P\sigma}{\gamma - \sigma}\right),$$

$$\dot{l}T = \frac{(\Delta + \kappa_m)\sigma}{2(\gamma - \sigma)} + \tfrac{2}{3}P\frac{\gamma\sigma}{2(\gamma - \sigma)^2}(\Delta + \kappa_m).$$

Let

$$p_m = \frac{2\gamma - 2\sigma}{2\gamma - 2\sigma}, \qquad p_{,,} = \frac{2\gamma}{2\gamma - 2\sigma}, \qquad p_s = \frac{2\gamma - 2\eta}{2\gamma - 2\sigma},$$

$$p_o = \frac{\gamma - 2\sigma}{2\gamma - 2\sigma}, \qquad p_{,} = \frac{\gamma}{2\gamma - 2\sigma}, \qquad p_p = \frac{\gamma - 2\eta}{2\gamma - 2\sigma},$$

and let

$$\epsilon = \tfrac{2}{3}\frac{\gamma\sigma}{2(\gamma - \sigma)^2} = \tfrac{1}{3}p_{,,}(p_{,,} - 1) = \cdot 01311.$$

Then if l_o, $l_{o,}$ be the moon's and sun's longitudes at the time of fictitious transit, we have

$$l = l_o + \frac{\sigma}{2\gamma - 2\sigma}(\Delta + \kappa_m) + \epsilon P(\Delta + \kappa_m),$$

$$l_, = l_{o,} + \frac{\eta}{2\gamma - 2\sigma}(\Delta + \kappa_m) + \epsilon\frac{\eta}{\gamma}P(\Delta + \kappa_m).$$

Then neglecting the term in P when multiplied by the small fraction η/γ, and introducing the p's, we have

$$\left.\begin{aligned} 2l &= 2l_o + (p_{,,} - 1)(\Delta + \kappa_m) + 2\epsilon P(\Delta + \kappa_m), \\ 2(l - l_,) &= 2(l_o - l_{o,}) + (p_s - 1)(\Delta + \kappa_m) + 2\epsilon P(\Delta + \kappa_m), \\ -l &= -l_o + (p_o - \tfrac{1}{2})(\Delta + \kappa_m) - \epsilon P(\Delta + \kappa_m), \\ l &= l_o + (p_, - \tfrac{1}{2})(\Delta + \kappa_m) + \epsilon P(\Delta + \kappa_m), \\ l - 2l_, &= l_o - 2l_{o,} + (p_p - \tfrac{1}{2})(\Delta + \kappa_m) + \epsilon P(\Delta + \kappa_m). \end{aligned}\right\} \quad . \quad . \quad (24).$$

We must now take the several terms of the expressions for the tide in (17) (18) (19), and rearrange them by aid of (24).

(i.) $\cos(\Delta + \kappa_m + 2l - \kappa_{,,}) + \cdot 683\, P\cos(\Delta + \kappa_m + 2l - \kappa_{,,})$

$$= \cos[p_{,,}(\Delta + \kappa_m) + 2l_o - \kappa_{,,}] + \cdot 683\, P\cos[{}_{,,}p(\Delta + \kappa_m) + 2l_o - \kappa_{,,}],$$

where

$${}_{,,}p = p_{,,} + \frac{2\epsilon}{\cdot 683} = p_{,,} + \cdot 0384 = 1\cdot 0379 + \cdot 0384 = 1\cdot 0763.$$

(ii.) $\cos(\Delta + \kappa_m + 2(l - l_,) - \kappa_s) = \cos[p_s(\Delta + \kappa_m) + 2(l_o - l_{o,}) - \kappa_s]$

$$- 2\epsilon P(\Delta + \kappa_m)\sin[p_s(\Delta + \kappa_m) + 2(l_o - l_{o,}) - \kappa_s].$$

Now the maximum value of P is about $\frac{1}{6}$, and $\Delta + \kappa_m$ will not differ largely from κ_m, because Δ will oscillate about the value zero, and κ_m might always be taken as less than 180°, either positively or negatively. Hence the coefficient of this second term cannot exceed $2 \times \cdot 013 \times \frac{1}{6} \times \pi$, or about $\cdot 013$.

Thus the second term cannot be more than about an eightieth of the first, and may be neglected. This term may also be safely neglected, even when κ_m is greater than 180°.

(iii.) $$\cos[\tfrac{1}{2}(\Delta+\kappa_m)-l+\tfrac{1}{2}\pi-\kappa_o]+P\beta\cos[\tfrac{1}{2}(\Delta+\kappa_m)-l+\tfrac{1}{2}\pi-{}_o\kappa]$$
$$=\cos[p_o(\Delta+\kappa_m)-l_o+\tfrac{1}{2}\pi-\kappa_o]+P\beta\cos[{}_op(\Delta+\kappa_m)-l_o+\tfrac{1}{2}\pi-{}_o\kappa],$$
where
$${}_op=p_o-\frac{\epsilon}{\beta}=\cdot 4811-\frac{\cdot 0131}{\beta}.$$

The reader who verifies the above formula will perceive the nature of the approximation adopted.

(iv.) $$\cos[\tfrac{1}{2}(\Delta+\kappa_m)+l-\tfrac{1}{2}\pi-\kappa_,]+\cdot 683\,P\cos[\tfrac{1}{2}(\Delta+\kappa_m)+l-\tfrac{1}{2}\pi-\kappa_,]$$
$$=\cos[p_,(\Delta+\kappa_m)+l_o-\tfrac{1}{2}\pi-\kappa_,]+\cdot 683\,P\cos[{}_,p(\Delta+\kappa_m)+l_o-\tfrac{1}{2}\pi-\kappa_,],$$
where
$${}_,p=p_,+\frac{\epsilon}{\cdot 683}=p_,+\cdot 0192=\cdot 5189+\cdot 0192=\cdot 5381.$$

(v.) The P term may be written
$$\cos[p_p(\Delta+\kappa_m)+l_o-2l_{o,}+\tfrac{1}{2}\pi-\kappa_p],$$
with a similar neglect to that involved in (ii.), but, of course, less important.

Before using these transformations the following notation must be introduced
$$\left.\begin{aligned}☽&=l_o-l_{o,},\\ \odot&=l_{o,}+5^\circ.\end{aligned}\right\}\qquad\qquad(25).$$

So that $l_o=☽+\odot-5^\circ$, $l_{o,}=\odot-5^\circ$. We then write

$$\begin{aligned}
\vartheta_m&=p_m\kappa_m-\kappa_m,\\
{}_m\vartheta&=p_m\kappa_m-{}_m\kappa \text{ (see (20), § 3)}\\
\vartheta_{,,}&=2(☽+\odot-5^\circ)+p_{,,}\kappa_m-\kappa_{,,},\\
{}_{,,}\vartheta&=\vartheta_{,,}+({}_{,,}p-p_{,,})\kappa_m=\vartheta_{,,}+\cdot 0384\,\kappa_m,\\
\vartheta_s&=2☽+p_s\kappa_m-\kappa_s,\\
\vartheta_o&=-(☽+\odot-5^\circ)+p_o\kappa_m-\kappa_o+90^\circ,\\
{}_o\vartheta&=-(☽+\odot-5^\circ)+p_o\kappa_m-{}_o\kappa-\frac{\cdot 01311}{\beta}\kappa_m+90^\circ \text{ (see (20), § 3)},\\
\vartheta_,&=(☽+\odot-5^\circ)+p_,\kappa_m-\kappa_,-90^\circ,\\
{}_,\vartheta&=\vartheta_,+({}_,p-p_,)\kappa_m=\vartheta_,+\cdot 0192\,\kappa_m,\\
\vartheta_p&=(☽-\odot+5^\circ)+p_p\kappa_m-\kappa_p+90^\circ\qquad\qquad(26).
\end{aligned}$$

We also require for the quater-diurnal tides

$$\vartheta_{2m} = 2p_m\kappa_m - \kappa_{2m},$$
$$\vartheta_{2s} = 2p_s\kappa_m - \kappa_{2s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

and for the annual and semi-annual tides

$$\vartheta_{sa} = (\odot - 5^\circ) - \kappa_{sa},$$
$$\vartheta_{ssa} = 2(\odot - 5^\circ) - \kappa_{ssa} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

The parts of the ϑ's which are independent of ☽ and ⊙ will, in the example, be written K, with appropriate suffix.

The moon's mean parallax is 57′, then when its parallax is $57' + \Pi$,

$$1 + P = (1 + \tfrac{1}{57}\Pi)^3,$$

so that $P = \frac{1}{19}\Pi = \cdot0526\,\Pi$ approximately.

Π will be substituted for P, and the numerical coefficients in h_P will be divided by 19 or multiplied by ·0526.

We observe then that $\frac{1}{19} \times \cdot683 = \cdot0360$, and $\frac{1}{19}\alpha = \alpha'$, $\frac{1}{19}\beta = \beta'$.

We can, by aid of the transformations (i.) to (v.) and the formulæ (26), now re-write (17) (18) (19), but some part of what was originally included in h is transferred to h_P. The result is as follows :—

$$\begin{aligned} h = {} & \mathrm{H}_m \cos(p_m\Delta + \vartheta_m) + \mathrm{H}_{,,} \cos(p_{,,}\Delta + \vartheta_{,,}) + \mathrm{H}_s \cos(p_s\Delta + \vartheta_s) \\ & \pm \mathrm{H}_o \cos(p_o\Delta + \vartheta_o) \pm \mathrm{H}_{,} \cos(p_{,}\Delta + \vartheta_{,}) \pm \mathrm{H}_p \cos(p_p\Delta + \vartheta_p) \\ & + \mathrm{H}_{2m} \cos(2p_m\Delta + \vartheta_{2m}) + \mathrm{H}_{2s} \cos(2p_s\Delta + \vartheta_{2s}) \quad . \quad . \quad . \quad (27). \end{aligned}$$

$$\begin{aligned} h_P = \Pi\,\{ & \alpha' \mathrm{H}_m \cos(p_m\Delta + {}_m\vartheta) + \cdot0360\,\mathrm{H}_{,,} \cos({}_{,,}p\Delta + {}_{,,}\vartheta) \\ & \pm \beta' \mathrm{H}_o \cos({}_op\Delta + {}_o\vartheta) \pm \cdot0360\,\mathrm{H}_{,} \cos({}_{,}p\Delta + {}_{,}\vartheta)\} \quad . \quad . \quad . \quad (28). \end{aligned}$$

$$\begin{aligned} h_{\Omega} = {} & \cos\Omega\,\{-\mathrm{H}_m \cdot037 \cos(p_m\Delta + \vartheta_m) + \mathrm{H}_{,,} \cdot283 \cos(p_{,,}\Delta + \vartheta_{,,}) \\ & \pm \mathrm{H}_o \cdot188 \cos(p_o\Delta + \vartheta_o) \pm \mathrm{H}_{,} \cdot115 \cos(p_{,}\Delta + \vartheta_{,})\} \\ & + \sin\Omega\,\{\mathrm{H}_m \cdot037 \sin(p_m\Delta + \vartheta_m) + \mathrm{H}_{,,} \cdot308 \sin(p_{,,}\Delta + \vartheta_{,,}) \\ & \mp \mathrm{H}_o \cdot188 \sin(p_o\Delta + \vartheta_o) \pm \mathrm{H}_{,} \cdot154 \sin(p_{,}\Delta + \vartheta_{,})\} \quad . \quad (29). \end{aligned}$$

Annual and semi-annual tides $= \mathrm{H}_{sa} \cos\vartheta_{sa} + \mathrm{H}_{ssa} \cos\vartheta_{ssa}$ . . . . (30).

Except for the solar parallactic portion to be considered later, this is the final expression for the tide. Apart from the slow variability of Π and Ω the only variable is Δ, for the ϑ's are not continuous variables, but change *per saltum* from one lunar transit to the next. Thus, in finding the maximum or H. W., we may treat Δ as the variable.

§ 6. *The Time and Height of High Water.*

Apart from parallactic and nodal corrections H. W. occurs when h in (27) is a maximum, and the value of Δ which satisfies that condition will give the time estimated from the epoch when Δ is zero. The value of Δ, when reduced to time in the manner shown below, gives the inequality in the interval after moon's transit, and will be called I. The time from moon's transit until Δ is zero is the mean interval, and will be called i. Then the interval from moon's transit is $i + I$. There is indeed a small parallactic correction to i, which is omitted in this statement, but is included below.

I begin by considering the mean interval.

Δ vanishes when $2(t + h - l) - \kappa_m = 0$, that is to say, at a time $\kappa_m/2(\gamma - \dot{l})$ after moon's transit.

Now, as shown in § 5,

$$\gamma - \dot{l} = (\gamma - \sigma)\left(1 + \tfrac{2}{3}\frac{P\sigma}{\gamma - \sigma}\right);$$

therefore

$$\frac{\kappa_m}{2(\gamma - \dot{l})} = \frac{\kappa_m}{2(\gamma - \sigma)} + \frac{\kappa_m}{2(\gamma - \sigma)}\,\frac{2}{3}\,\frac{P\sigma}{\gamma - \sigma}.$$

The first of these terms is denoted by i, so that

$$i = \frac{\kappa_m}{2(\gamma - \sigma)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The second term is the parallactic correction to the mean interval, and will be considered in § 8, together with the other parallactic corrections.

If we denote by the suffixes 2, 1, 4 the semi-diurnal, diurnal, and quater-diurnal terms, (27) may be written

$$h = \Sigma\, \mathrm{H}_2 \cos(p_2\Delta + \vartheta_2) + \Sigma\, \mathrm{H}_1 \cos(p_1\Delta + \vartheta_1) + \Sigma\, \mathrm{H}_4 \cos(p_4\Delta + \vartheta_4).$$

The condition for maximum is that $dh/d\Delta$ should vanish, and the equation consists of the sum of three pairs of terms of the form $\Sigma\,[\mathrm{H}p \cos\vartheta \sin p\Delta + \mathrm{H}p \sin\vartheta \cos p\Delta]$ equated to zero.

It is well known that

$$\frac{\sin k\alpha}{k \sin\alpha} = 1 + \frac{1 - k^2}{3!}\sin^2\alpha + \ldots$$

$$\frac{\cos k\alpha}{\cos\alpha} = 1 + \frac{1 - k^2}{2!}\sin^2\alpha + \ldots$$

Now all the p_2's are nearly unity, the p_1's nearly $\frac{1}{2}$, the p_4's nearly 2. Hence to a near approximation

$$\sin p_n\Delta = \frac{2p_n}{n}\sin\tfrac{1}{2}n\Delta\left[1 - \tfrac{1}{6}\left(\frac{4p_n^2}{n^2} - 1\right)\sin^2\tfrac{1}{2}n\Delta\right],$$

$$\cos p_n \quad = \cos\tfrac{1}{2}n\Delta\left[1 - \tfrac{1}{2}\left(\frac{4p_n^2}{n^2} - 1\right)\sin^2\tfrac{1}{2}n\Delta\right],$$

where n is 2, 1, or 4.

The p's are so nearly equal to 1, $\frac{1}{2}$, 2 respectively, that the second terms are small. In the case of the second term of $\sin p_1\Delta$, $\cos p_1\Delta$ we have the factor $\sin^2\Delta$, which is equal to $\frac{1}{2}$ even when Δ is 90°; also the height of the quater-diurnal tide is small. Thus, both for the diurnal and quater-diurnal tides, the second term may be neglected.

In the case of the semi-diurnal terms, indeed, the correction is so small that it may certainly be neglected if Δ be less than 45° or 50°, and without much loss of accuracy even for larger values of Δ. We shall in the first place neglect these second terms entirely.

Now let

$$\left.\begin{aligned} F &= \Sigma\, Hp\sin\vartheta \\ G &= \Sigma\, Hp^2\cos\vartheta \end{aligned}\right\} \quad \ldots \ldots \quad (32),$$

with suffixes 2, 1, 4 for semi-diurnal, diurnal, quater-diurnal terms.

The constituent terms of F, G will be written separately below, for example $F_s = H_s p_s \sin\vartheta_s$.

Then the equation $dh/d\Delta = 0$, leads to

$$F_2\cos\Delta + G_2\sin\Delta \pm (F_1\cos\tfrac{1}{2}\Delta + 2G_1\sin\tfrac{1}{2}\Delta) + F_4\cos 2\Delta + \tfrac{1}{2}G_4\sin 2\Delta = 0. \quad (33).$$

With regard to the additional terms referred to above, the values of the semi-diurnal p's are, $p_m = 1$, $p_{//} = 1{\cdot}038$, $p_s = 1{\cdot}035$, and hence $p_{//}^2 - 1$ and $p_s^2 - 1$ are both nearly equal to ·072. Therefore if (33) be regarded as the fundamental equation, the additional terms may be taken into account by supposing that there are corrections to F_2 and G_2 given by

$$\left.\begin{aligned} \delta F_2 &= -{\cdot}012\sin^2\Delta\,.\,3F_2 \\ \delta G_2 &= -{\cdot}012\sin^2\Delta\,(G_{//} + G_s) \end{aligned}\right\} \quad \ldots \ldots \quad (34).$$

The equation (33) may be solved thus :—
take

$$\tan\Delta_0 = -\frac{F_2 \pm F_1}{G_2 \pm G_1} \quad \ldots \ldots \quad (35).$$

Then if we put

$$D_o = F_2 \sin \Delta_o - G_2 \cos \Delta_o \pm \tfrac{1}{2} (F_1 \sin \tfrac{1}{2} \Delta_o - 2G_1 \cos \tfrac{1}{2} \Delta_o) + 2 (F_4 \sin 2\Delta_o - \tfrac{1}{2} G_4 \cos 2 \Delta_o)$$

$$N_o = F_2 \cos \Delta_o + G_2 \sin \Delta_o \pm (F_1 \cos \tfrac{1}{2} \Delta_o + 2G_1 \sin \tfrac{1}{2} \Delta_o) + (F_4 \cos 2\Delta_o + \tfrac{1}{2} G_4 \cos 2\Delta_o) \quad . \quad . \quad . \quad (36),$$

and

$$\delta\Delta = \frac{N_o}{D_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

the solution of the equation (33) is $\Delta = \Delta_o + \delta\Delta$; and if D is the value of D_o when Δ replaces Δ_o, it is clear that

$$\delta D = D - D_o = \delta\Delta \{(F_2 \cos \Delta_o + G_2 \sin \Delta_o) \pm \tfrac{1}{4} (F_1 \cos \tfrac{1}{2} \Delta_o + 2G_1 \sin \tfrac{1}{2} \Delta_o) + 4 (F_4 \cos 2\Delta_o + \tfrac{1}{2} G_4 \sin 2\Delta_o)\} \quad . \quad . \quad (38).$$

The angle Δ has to be reduced to time by division by $2(\gamma - \dot{l})$. As in the case of the reduction of the mean interval

$$\frac{\Delta}{2(\gamma - \dot{l})} = \frac{\Delta}{2(\gamma - \sigma)} \left(1 + \tfrac{2}{3} \frac{P\sigma}{\gamma - \sigma}\right).$$

Now the greatest value of P is about $\frac{1}{5}$, and $\sigma/(\gamma - \sigma)$ is ·038, hence $\frac{2}{3} P\sigma/(\gamma - \sigma)$ is at greatest ·005; but the inequality in the interval is rarely more than 2^h, and even if $\Delta/2(\gamma - \sigma)$ amounted to 3^h, the second term would only be about 1^m. Therefore I neglect the parallactic inequality in the reduction of Δ to time, and simply divide Δ by $2(\gamma - \sigma)$.

The value of $2(\gamma - \sigma)$ is 28°·98 per hour, and the reciprocal of this is ·0345, or $\frac{1}{30} + \frac{21}{20} \cdot \frac{1}{30^2}$.

Hence, if Δ be given in degrees, we have to multiply by ·0345, or its equivalent, to find its value in hours.

If any correction be given in circular measure the reduction to time is also very simple, for $57°·296 \times ·0345 = 1^h·977 = 119^m$, or

$$\frac{1}{2(\gamma - \sigma)} \text{(circ. meas.)} = 119^m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39).$$

Thus, we have for H. W.,

$$\left.\begin{aligned} \Delta &= \Delta_o + \delta\Delta \\ D &= D_o + \delta D \\ I &= ·0345\, \Delta = \left(\tfrac{1}{30} + \tfrac{21}{20} \cdot \tfrac{1}{30^2}\right) \Delta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40).$$

Turning now to the expression for the height: it is expressible as the sum of three terms of the form

$$\Sigma\, (H \cos \vartheta \cos p\Delta - H \sin \vartheta \sin p\Delta).$$

Let

$$\left.\begin{array}{l} A = \Sigma\, H \cos \vartheta, \\ \text{then} \\ h = A_2 \cos \Delta - F_2 \sin \Delta \pm (A_1 \cos \tfrac{1}{2}\Delta - 2F_1 \sin \tfrac{1}{2}\Delta) + A_4 \cos 2\Delta - \tfrac{1}{2} F_4 \sin 2\Delta \end{array}\right\} \quad (41),$$

with a correction to the height corresponding with

$$\left.\begin{array}{l} \delta A_2 = -\,{\cdot}012 \sin^2 \Delta \,.\, 3\,(A_{\prime\prime} + A_s) \\ \delta F_2 = -\,{\cdot}012 \sin^2 \Delta \,.\, F_2 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42).$$

In §§ 8, 9 we shall have to consider the variations of the interval and height due to variations of semi-diurnal and diurnal A, F, G. It is clear from the preceding formulæ and from (39) that

$$\left.\begin{array}{l} \delta I = \dfrac{119^m}{D} \{\delta F_2 \cos \Delta + \delta G_2 \sin \Delta \pm (\delta F_1 \cos \tfrac{1}{2}\Delta + 2\,\delta G_1 \sin \tfrac{1}{2}\Delta)\} \\ \delta h = \delta A_2 \cos \Delta - \delta F_2 \sin \Delta \pm (\delta A_1 \cos \tfrac{1}{2}\Delta - 2\,\delta F_1 \sin \tfrac{1}{2}\Delta) \end{array}\right\} \quad (43).$$

A particular case of the application of (43) is to the computation of the corrections referred to in (34) and (42).

§ 7. *On Evanescent Tides.*

At certain parts of the lunation, the diurnal tides sometimes suffice to annul one H. W. and one L. W., so that there is only one tide a day, perhaps for several days running.

If the inferior H. W. be watched as the condition of evanescence approaches, it will be seen to become smaller and smaller, and to occur later and later or earlier and earlier, and the adjacent L. W. undergoes similar changes. In the limit H. W. and L. W. coalesce, and in a tide diagram the coalescence appears as a point of contrary reflexure with horizontal tangent. Beyond this point, the reflexure is still maintained, although the tangent is not horizontal; finally, the tangent again becomes horizontal, and the double H. W. and L. W. again reappear.

Now in the use of the method of this paper, the loss of the double tide is very inconvenient, and I therefore propose to take the point of reflexure as representing both H. W. and L. W. during evanescence.

If N cannot vanish there is evanescence, and the point of reflexure is given by D = 0. The limit of the H. W. is given by N = 0, D = 0 simultaneously, and beyond this only D can vanish. The vanishing of D is taken to represent H. W.

Accordingly, when N cannot vanish, we proceed to make D vanish thus :—

$$0 = F_2 \sin \Delta - G_2 \cos \Delta \pm \tfrac{1}{2} (F_1 \sin \tfrac{1}{2} \Delta - 2 G_1 \cos \tfrac{1}{2} \Delta) + 2 (F_4 \sin 2\Delta - \tfrac{1}{2} G_4 \cos 2\Delta) \quad . \quad . \quad (44).$$

As a first approximation put

$$\tan \Delta_0 = \frac{G_2 \pm G_1}{F_2 \pm \frac{1}{4} F_1}$$

Then writing

$$E_o = - \{F_2 \cos \Delta_o + G_2 \sin \Delta_o \pm \tfrac{1}{4} (F_1 \cos \tfrac{1}{2} \Delta_o + 2G_1 \sin \tfrac{1}{2}\Delta_o) + 4 (F_4 \cos 2\Delta_o + \tfrac{1}{2} G_4 \sin 2\Delta_o)\} \quad . \quad . \quad . \quad . \quad (45),$$

we have

$$\Delta = \Delta_o + \delta\Delta, \text{ where } \delta\Delta = \frac{D_o}{E_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46).$$

In the rest of the calculation this value of Δ is to be treated exactly as though it had been determined by the former method.

The corrections M, N, P, Q, R, S considered in succeeding sections, however, present a difficulty. In this case, Δ will always be very nearly $\pm 90°$, and I propose to compute P, Q, S (see §§ 8, 9), as though that were the true value of Δ.

But the correctional terms M′, N′, R, defined in §§ 8, 9, become theoretically infinite, and we are therefore compelled not to compute them, and to fill up the hiatus in the manner shown in the example below.

The process here suggested is a makeshift, but it is sufficient for the construction of a trustworthy tide-table, since the real occurrence at these times is a long period of nearly slack water, with or without a small maximum and minimum.

§ 8. *Parallactic Corrections to Time and Height.*

We will first consider the parallactic correction to the mean interval; we saw at the beginning of § 6 that there is a correction to the interval due to the moon's unequal motion in longitude of

$$\frac{\kappa_m}{2(\gamma - \sigma)} \cdot \frac{2}{3} \frac{P\sigma}{\gamma - \sigma} .$$

Now according to (31) $\kappa_m/2(\gamma - \sigma)$ is i, the mean interval, which we will suppose expressed in hours; also P is $\frac{1}{19}\Pi$ and $\sigma/(\gamma - \sigma)$ is ·03788, and $\frac{2}{3}$ of 60^m is 40^m; hence the correction in minutes is

$$i \,.\, 40^m \,.\, {\cdot}03788 \,.\, \tfrac{1}{19} \,.\, \Pi = i \times 0^m{\cdot}0797\, \Pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (47).$$

Now turning to the other parallactic terms :—

Let us introduce the following notation, which follows the same plan as that used before, viz. :—

$$\left.\begin{matrix} {}_m\text{A} \\ {}_m\text{F} \\ {}_m\text{G} \end{matrix}\right\} = \alpha'\text{H}_m \left.\begin{matrix} & \cos \\ p_m & \sin \\ p_m^2 & \cos \end{matrix}\right\} {}_m\vartheta, \qquad \left.\begin{matrix} {}_{\prime\prime}\text{A} \\ {}_{\prime\prime}\text{F} \\ {}_{\prime\prime}\text{G} \end{matrix}\right\} = \cdot 036\ \text{H}_{\prime\prime} \left.\begin{matrix} & \cos \\ {}_{\prime\prime}p & \sin \\ {}_{\prime\prime}p^2 & \cos \end{matrix}\right\} {}_{\prime\prime}\vartheta,$$

$$\left.\begin{matrix} {}_o\text{A} \\ {}_o\text{F} \\ {}_o\text{G} \end{matrix}\right\} = \beta'\text{H}_o \left.\begin{matrix} & \cos \\ {}_op & \sin \\ {}_op^2 & \cos \end{matrix}\right\} {}_o\vartheta, \qquad \left.\begin{matrix} {}_{\prime}\text{A} \\ {}_{\prime}\text{F} \\ {}_{\prime}\text{G} \end{matrix}\right\} = \cdot 036\ \text{H}_{\prime} \left.\begin{matrix} & \cos \\ {}_{\prime}p & \sin \\ {}_{\prime}p^2 & \cos \end{matrix}\right\} {}_{\prime}\vartheta.$$

Then a comparison of the method of § 6 with (28) § 5 shows that the corrections to the time and height are to be found by applying corrections to the A's, F's, G's, as follows :

$$\delta\text{F}_2 = \Pi\,({}_m\text{F} + {}_{\prime\prime}\text{F}) = \Pi l_2, \quad \delta\text{G}_2 = \Pi\,({}_m\text{G} + {}_{\prime\prime}\text{G}) = \Pi m_2,$$
$$\delta\text{F}_1 = \Pi\,({}_o\text{F} + {}_{\prime}\text{F}) = \Pi l_1, \quad \delta\text{G}_1 = \Pi\,({}_o\text{G} + {}_{\prime}\text{G}) = \Pi m_1,$$
$$\delta\text{A}_2 = \Pi\,({}_m\text{A} + {}_{\prime\prime}\text{A}) = \Pi z_2,$$
$$\delta\text{A}_1 = \Pi\,({}_o\text{A} + {}_{\prime}\text{A}) = \Pi z_1 \quad \ldots \ldots \quad (48).$$

Then comparing (47) (48) with (39) and (43) we see that if

$$\text{R} = 119^{\text{m}} \frac{1}{\text{D}} \{l_2 \cos\Delta + m_2 \sin\Delta \pm (l_1 \cos\tfrac{1}{2}\Delta + 2m_1 \sin\tfrac{1}{2}\Delta)\} + 0^{\text{m}}\cdot 08\,.\,i$$
$$\text{S} = \{z_2 \cos\Delta - l_2 \sin\Delta \pm (z_1 \cos\tfrac{1}{2}\Delta - 2l_1 \sin\tfrac{1}{2}\Delta)\} \quad \ldots \quad (49),$$

the corrections to the interval and height are

$$\left.\begin{matrix} \delta I_{\text{P}} = \Pi\text{R}, \\ \delta h_{\text{P}} = \Pi\text{S}, \end{matrix}\right\} \quad \ldots \ldots \quad (50).$$

§ 9. *Corrections to Time and Height for Longitude of Moon's Node.*

Here again, we treat the terms as corrections to A, F, G.

Let

$$
\begin{aligned}
c_2 &= & & \cdot 283\,\mathrm{H}_{\prime\prime}p_{\prime\prime}\sin\vartheta_{\prime\prime} &&= & & \cdot 283\,\mathrm{F}_{\prime\prime},\\
d_2 &= -\cdot 037\,\mathrm{H}_m p_m\cos\vartheta_m & & -\cdot 308\,\mathrm{H}_{\prime\prime}p_{\prime\prime}\cos\vartheta_{\prime\prime} &&= -\cdot 037\,\mathrm{G}_m & & -\cdot 296\,\mathrm{G}_{\prime\prime},\\
e_2 &= -\cdot 037\,\mathrm{H}_m p_m^2\cos\vartheta_m & & +\cdot 283\,\mathrm{H}_{\prime\prime}p_{\prime\prime}^2\cos\vartheta_{\prime\prime} &&= -\cdot 037\,\mathrm{G}_m & & +\cdot 283\,\mathrm{G}_{\prime\prime},\\
f_2 &= & & \cdot 308\,\mathrm{H}_{\prime\prime}p_{\prime\prime}^2\sin\vartheta_{\prime\prime} &&= & & \cdot 319\,\mathrm{F}_{\prime\prime},\\
a_2 &= -\cdot 037\,\mathrm{H}_m\cos\vartheta_m & & +\cdot 283\,\mathrm{H}_{\prime\prime}\cos\vartheta_{\prime\prime} &&= -\cdot 037\,\mathrm{A}_m & & +\cdot 283\,\mathrm{A}_{\prime\prime},\\
b_2 &= & & \cdot 308\,\mathrm{H}_{\prime\prime}\sin\vartheta_{\prime\prime} &&= & & \cdot 296\,\mathrm{F}_{\prime\prime},\\
c_1 &= \cdot 188\,\mathrm{H}_o p_o\sin\vartheta_o & & +\cdot 115\,\mathrm{H}_{\prime}p_{\prime}\sin\vartheta_{\prime} &&= \cdot 188\,\mathrm{F}_o & & +\cdot 115\,\mathrm{F}_{\prime},\\
d_1 &= \cdot 188\,\mathrm{H}_o p_o\cos\vartheta_o & & -\cdot 154\,\mathrm{H}_{\prime}p_{\prime}\cos\vartheta_{\prime} &&= \cdot 391\,\mathrm{G}_o & & -\cdot 297\,\mathrm{G}_{\prime},\\
e_1 &= \cdot 188\,\mathrm{H}_o p_o^2\cos\vartheta_o & & +\cdot 115\,\mathrm{H}_{\prime}p_{\prime}^2\cos\vartheta_{\prime} &&= \cdot 188\,\mathrm{G}_o & & +\cdot 115\,\mathrm{G}_{\prime},\\
f_1 &= -\cdot 188\,\mathrm{H}_o p_o^2\sin\vartheta_o & & +\cdot 154\,\mathrm{H}_{\prime}p_{\prime}^2\sin\vartheta_{\prime} &&= -\cdot 0905\,\mathrm{F}_o & & +\cdot 080\,\mathrm{F}_{\prime},\\
a_1 &= \cdot 188\,\mathrm{H}_o\cos\vartheta_o & & +\cdot 115\,\mathrm{H}_{\prime}\cos\vartheta_{\prime} &&= \cdot 188\,\mathrm{A}_o & & +\cdot 115\,\mathrm{A}_{\prime},\\
b_1 &= -\cdot 188\,\mathrm{H}_o\sin\vartheta_o & & +\cdot 154\,\mathrm{H}_{\prime}\sin\vartheta_{\prime} &&= -\cdot 391\,\mathrm{F}_o & & +\cdot 297\,\mathrm{F}_{\prime}. \quad (51).
\end{aligned}
$$

A comparison with (29) then, shows that

$$
\begin{aligned}
\delta\mathrm{F}_2 &= c_2\cos\Omega + d_2\sin\Omega, & \delta\mathrm{F}_1 &= c_1\cos\Omega + d_1\sin\Omega,\\
\delta\mathrm{G}_2 &= e_2\cos\Omega + f_2\sin\Omega, & \delta\mathrm{G}_1 &= e_1\cos\Omega + f_1\sin\Omega,\\
\delta\mathrm{A}_2 &= a_2\cos\Omega + b_2\sin\Omega, & \delta\mathrm{A}_1 &= a_1\cos\Omega + b_1\sin\Omega \quad . \quad . \quad . \quad (52).
\end{aligned}
$$

Then putting

$$
\begin{aligned}
\mathrm{M}' - \delta\mathrm{M} &= 119^{\mathrm{m}}\frac{1}{\mathrm{D}}\{c_2\cos\Delta + e_2\sin\Delta \pm (c_1\cos\tfrac{1}{2}\Delta + 2e_1\sin\tfrac{1}{2}\Delta)\},\\
\mathrm{N}' - \delta\mathrm{N} &= 119^{\mathrm{m}}\frac{1}{\mathrm{D}}\{d_2\cos\Delta + f_2\sin\Delta \pm (d_1\cos\tfrac{1}{2}\Delta + 2f_1\sin\tfrac{1}{2}\Delta)\},\\
\mathrm{P}' &= a_2\cos\Delta - c_2\sin\Delta \pm (a_1\cos\tfrac{1}{2}\Delta - 2c_1\sin\tfrac{1}{2}\Delta),\\
\mathrm{Q}' &= b_2\cos\Delta - d_2\sin\Delta \pm (b_1\cos\tfrac{1}{2}\Delta - 2d_1\sin\tfrac{1}{2}\Delta) \quad . \quad (53).
\end{aligned}
$$

The corrections to interval and height, as far as concerns the investigation up to the present point, are

$$
\begin{aligned}
\delta I_{\Omega} &= (\mathrm{M}' - \delta\mathrm{M})\cos\Omega + (\mathrm{N}' - \delta\mathrm{N})\sin\Omega,\\
\delta h_{\Omega} &= \mathrm{P}'\cos\Omega + \mathrm{Q}'\sin\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54).
\end{aligned}
$$

§ 10. *Reference to the Moon's True Transit.*

The intervals have been referred to the transits of a fictitious satellite whose R. A. is equal to l, and we now require corrections so as to refer to the moon's true transit.

Let T, T′ be the times of true and fictitious transits, and let α be the moon's R. A. Then if t denotes the mean solar hour angle at time T,

$$t + h - \alpha = q\pi,$$

and, dropping the suffix o to l, at time T′

$$t + h - l = q\pi,$$

where q is an even integer at upper, and an odd one at lower transit.

If α be the value of the moon's R. A. at time T′, then its value at T is $\alpha + \dot{\alpha}(\mathrm{T} - \mathrm{T}')$; also the $t + h$ of the first equation is equal to the $t + h$ of the second corrected by $\gamma(\mathrm{T} - \mathrm{T}')$.

Hence the two equations may be written

$$t + h + \gamma(\mathrm{T} - \mathrm{T}') - \alpha - \dot{\alpha}(\mathrm{T} - \mathrm{T}') = q\pi,$$
$$t + h - l = q\pi.$$

Hence

$$\mathrm{T} - \mathrm{T}' = \frac{\alpha - l}{\gamma - \dot{\alpha}}.$$

It will afford a sufficiently close approximation if we replace $\dot{\alpha}$ by the moon's mean motion σ, so that

$$\mathrm{T}' = \mathrm{T} + \frac{l - \alpha}{\gamma - \sigma}.$$

We have, therefore, to find the excess of the moon's longitude above her R. A. at the time of fictitious transit.

It will be seen from fig. 2 that we have to determine the relationship between the R. A. and longitude, both measured from the intersection of the orbit and equator. The formula is exactly analogous with that which gives the reduction of longitudes to the ecliptic, so that

$$l - \xi = \alpha - \nu + \tan^2 \tfrac{1}{2} I \sin 2(l - \xi) - \tfrac{1}{2} \tan^4 \tfrac{1}{2} I \sin 4(l - \xi) + \ldots.$$

The last of these terms is very small and may be neglected, so that

$$l - \alpha = -(\nu - \xi) + \tan^2 \tfrac{1}{2} I \cos 2\xi \sin 2l - \tan^2 \tfrac{1}{2} I \sin 2\xi \cos 2l.$$

By the formula (7)

$$\tan^2 \tfrac{1}{2} I = \tan^2 \tfrac{1}{2}\omega\,[1 + 2i \operatorname{cosec}\omega \cos \Omega],$$
$$\sin 2\xi = 2i \cot \omega \sin \Omega,$$
$$\cos 2\xi = 1,$$
$$\nu - \xi = i \tan \tfrac{1}{2}\omega \sin \Omega.$$

Hence

$$l - \alpha = \tan^2 \tfrac{1}{2}\,\omega \sin 2l + \cos \Omega \left[2i \frac{\tan^2 \frac{1}{2}\,\omega}{\sin \omega} \sin 2l \right]$$
$$+ \sin \Omega \left[- 2i \tan^2 \tfrac{1}{2}\,\omega \cot \omega \cos 2l - i \tan \tfrac{1}{2}\,\omega \right].$$

But $2l = 2 ☽ + 2 ☉ - 10° = \Theta$, suppose; and

$$\frac{\tan^2 \frac{1}{2}\,\omega}{\gamma - \sigma} = 0^{\text{h}}\cdot 172 = 10^{\text{m}}\cdot 32, \quad \frac{2i \tan^2 \frac{1}{2}\,\omega}{(\gamma - \sigma) \sin \omega} = 4^{\text{m}}\cdot 60, \quad \frac{2i \tan^2 \frac{1}{2}\,\omega \cot \omega}{\gamma - \sigma} = 4^{\text{m}}\cdot 22,$$

and $\dfrac{i \tan \frac{1}{2}\,\omega}{\gamma - \sigma} = 4^{\text{m}}\cdot 41.$

Let us put then

$$\left.\begin{aligned} \Theta &= 2 ☽ + 2 ☉ - 10°, \\ \delta \text{T} &= 0^{\text{h}}\cdot 172 \sin \Theta, \\ \delta \text{M} &= 4^{\text{m}}\cdot 60 \sin \Theta, \\ \delta \text{N} &= - \left[4^{\text{m}}\cdot 22 \cos \Theta + 4^{\text{m}}\cdot 41 \right], \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

and we have

$$\text{T}' = \text{T} + \delta \text{T} + \delta \text{M} \cos \Omega + \delta \text{N} \sin \Omega,$$

It is now clear that, when the tide is referred to the moon's true transit, δT must be added to I, and δM, δN must be added to M′ − δM, N′ − δN as given in (53) to find M′, N′.

Let 𝕬 be the height of mean sea level above the datum adopted for the tide-table, and let

$$\text{B}_o = \mathfrak{A} + \text{H}_{ssa} \cos \vartheta_{ssa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56).$$

Then i denotes the mean interval from moon's transit to H. W., and the interval is

$$i + I + \delta \text{T} + \text{M}' \cos \Omega + \text{N}' \sin \Omega + \delta I_{\text{P}};$$

and the height is

$$\text{B}_o + h + \text{P}' \cos \Omega + \text{Q}' \sin \Omega + \delta h_{\text{P}} + \text{H}_{sa} \cos \vartheta_{sa}.$$

In these formulæ all the quantities are functions of ☽; now although ☽ can easily be found from the 'Nautical Almanac,' it is not tabulated, and the difficulty of using the tide-table would be considerably augmented if it were necessary to compute ☽ for each tide. We shall proceed, then, to convert our formulæ into others, in which there is direct reference to moon's true transit, although there is some loss of accuracy in the process, and the amount of computation required to form the table is increased.

We have seen that the time T′ of fictitious transit is given by

$$\text{T}' = \text{T} + \delta \text{T} + \delta \text{M} \cos \Omega + \delta \text{N} \sin \Omega,$$

and that $(\gamma - \eta) T' + h - l = q\pi$ where h and l correspond to the time of fictitious transit.

Now

$$l_{,} = h + 2e_{,} \sin (l_{,} - 281^\circ),$$

where $e_{,}$ is the eccentricity of the sun's orbit, and 281° the longitude of perigee.

Also

$$l = \text{☽} + l_{,}.$$

Hence

$$l - h = \text{☽} + 2e_{,} \sin (l_{,} - 281^\circ).$$

But

$$\frac{2e_{,}}{\gamma - \eta} = 7^{m}{\cdot}69 = 0^{h}{\cdot}128;\ \text{also } l_{,} = \odot - 5^\circ, \text{and} - 286^\circ = + 74^\circ.$$

Therefore

$$T' = \frac{q\pi + \text{☽}}{\gamma - \eta} + 7^{m}{\cdot}69 \sin (\odot + 74^\circ), \text{ and}$$

$$\frac{q\pi + \text{☽}}{\gamma - \eta} = T + \delta T + \delta M \cos \text{☊} + \delta N \sin \text{☊} - 7^{m}{\cdot}69 \sin (\odot + 74^\circ).$$

This formula connects ☽ with T, the time of moon's transit.

Now I is a function of ☽ or of $(q\pi + \text{☽}) / (\gamma - \eta)$; hence if I (T) be the value of I for which $\text{☽} = (\gamma - \eta) T - q\pi$, and if I (☽) be the true value of I,

$$I(\text{☽}) = I(T) + \frac{dI}{dT} \{\delta T + \delta M \cos \text{☊} + \delta N \sin \text{☊} - 7^{m}{\cdot}69 \sin (\odot + 74^\circ)\};$$

and similarly

$$h(\text{☽}) = h(T) + \frac{dh}{dT} \{\delta T + \delta M \cos \text{☊} + \delta N \sin \text{☊} - 0^{h}{\cdot}128 \sin (\odot + 74^\circ)\}.$$

These expressions have now to be substituted in those for the height and interval; but in the small terms δT, M′, N′, &c., we may regard ☽ as denoting $(\gamma - \eta) T - q\pi$.

In carrying out the substitutions preparations will be made for computation.

First put

$$\delta I = \frac{dI}{dT} \delta T = \frac{dI}{dT} \times 10^{m}{\cdot}32 \sin \Theta,$$

$$= \left(2 \frac{dI}{dT}\right) \cdot (2{\cdot}34) (2^{m}{\cdot}2 \sin \Theta) \quad \dots \dots \dots \quad (57).$$

Similarly, since $2^{m}{\cdot}2$ is $\frac{11}{300}$ hrs., we put

$$\delta h = \left(2 \frac{dh}{dT}\right) (2{\cdot}34) (\tfrac{11}{300} \sin \Theta) \quad \dots \dots \dots \quad (58).$$

Next put

$$M'' = \frac{dI}{dT}\delta M = \frac{dI}{dT} \times 4^{m}\cdot 60 \sin \Theta,$$

$$= \left(2\frac{dI}{dT}\right)(2^{m}\cdot 2 \sin \Theta)(1 + \tfrac{1}{20}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

and

$$- N'' = - \frac{dI}{dT}\delta N,$$

$$= \left(2\frac{dI}{dT}\right)(2^{m}\cdot 2) + \left(2\frac{dI}{dT}\right)(2^{m}\cdot 2 \cos \Theta)(1 - \tfrac{1}{20}) \quad . \quad . \quad . \quad . \quad (60).$$

Similarly put

$$P'' = \left(2\frac{dh}{dT}\right)(\tfrac{11}{300} \sin \Theta)(1 + \tfrac{1}{20}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61),$$

$$- Q'' = \left(2\frac{dh}{dT}\right)(\tfrac{11}{300}) + \left(2\frac{dh}{dT}\right)(\tfrac{11}{300} \cos \Theta)(1 - \tfrac{1}{20}) \quad . \quad . \quad . \quad . \quad (62).$$

Lastly put

$$\mathfrak{i} = - \left(2\frac{dI}{dT}\right)\{3^{m}\cdot 85 \sin (\odot + 74^\circ)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

$$\mathfrak{h}' = - \left(2\frac{dh}{dT}\right)\{0^{h}\cdot 064 \sin (\odot + 74^\circ)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64).$$

With this notation

$$I(☽) = I(T) + \delta I + M'' \cos ☊ + N'' \sin ☊ + \mathfrak{i},$$
$$h(☽) = h(T) + \delta h + P'' \cos ☊ + Q'' \sin ☊ + \mathfrak{h}.$$

These are now to be substituted in the expressions for interval and height, and in doing so we may drop the (T) after the I and h.

I write

$$\mathfrak{I} = I + \delta T + i + \delta I,$$
$$\mathfrak{H} = B_o + h + \delta h,$$
$$\mathfrak{h} = \mathfrak{h}' + H_{sa} \cos \vartheta_{sa},$$
$$M = M' + M'', \quad N = N' + N'',$$
$$P^* = P' + P'', \quad Q = Q' + Q'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

Our formulæ are designed to serve for any time of year and its opposite; now since $\mathfrak{i}$ and $\mathfrak{h}$ involve the sun's longitude they change their signs in six months, and we write $\pm\,\mathfrak{i}$, $\pm\,\mathfrak{h}$, and it is to be understood that the upper sign is to be used for the time of year under computation and the lower for its opposite.

The interval is then

* This P will not be confused with the P defined in § 1, which has been replaced by II.

$$\mathfrak{I} \pm \mathfrak{i} + \mathrm{M} \cos \Omega + \mathrm{N} \sin \Omega + \mathrm{R}\, \Pi \qquad (66),$$

and the height is

$$\mathfrak{H} \pm \mathfrak{h} + \mathrm{P} \cos \Omega + \mathrm{Q} \sin \Omega + \mathrm{S}\, \Pi \qquad (67).$$

One part of M, N, P, Q arises from a true change in the tide when the longitude of the moon's node changes, and the remainder (nearly equally large) merely depends on the reference to true instead of fictitious transit. The quantities $\mathfrak{i}$ and $\mathfrak{h}$ depend partly on a portion of the equation of time and partly on the annual tide.

We must now explain the computation of $2\, dI/d\mathrm{T}$ and of $2\, dh/d\mathrm{T}$.

If $u_o, u_1, \ldots u_{23}$ are cyclical values of a function, the symmetrical interpolation formula in the neighbourhood of u_m is

$$u_{x+m} = u_m + \tfrac{1}{2} x (\Delta \mathrm{E}^{-1} + \Delta) u_m + \frac{x^2}{2!} \Delta^2 \mathrm{E}^{-1} u_m + \tfrac{1}{2} \frac{x(x^2-1)}{3!} (\Delta^3 \mathrm{E}^{-2} + \Delta^3 \mathrm{E}^{-1}) u_m + \ldots,$$

where

$$\mathrm{E} u_m = u_{m+1} \quad \text{and} \quad \Delta u_m = u_{m+1} - u_m.$$

Then when $x = 0$,

$$2 \frac{du_m}{dx} = (\Delta \mathrm{E}^{-1} + \Delta) u_m - \tfrac{1}{6} (\Delta^3 \mathrm{E}^{-2} + \Delta^3 \mathrm{E}^{-1}) u_m \ldots,$$

or

$$2 \frac{du_m}{dx} = (u_{m+1} - u_{m-1}) - \tfrac{1}{6} \Delta^2 \mathrm{E}^{-1} (u_{m+1} - u_{m-1}) + \ldots$$

In the present case the first term will usually suffice, but the second term may be easily computed.

In order then to compute the required differential coefficients we arrange the I's and h's in two columns, the even entries in one column and the odd in another, and take the differences of the two columns independently of one another.

§ 11. *The Correction for Solar Parallax.*

The terms depending on solar parallax arise from the potential $\mathrm{V}_{P,}$ in (15), but the correction is so small that I shall omit it from the example below. It is well, however, to show how it may be computed. The only terms of importance are those in H_s, $\mathrm{H}_,$ corresponding to the tides S_2, K_1.

The variability of $P_,$ enters into the calculation in the form of corrections to A_2, G_2, F_2, A_1, G_1, F_1.

The sun's parallax is approximately

$$1 + e_, \cos (l_, - 281^\circ), \text{ and hence } P_, = 3e_, \cos (\odot + 74^\circ).$$

Now

$$3e_, \text{ is } \cdot 0504, \text{ and } 119^{\text{m}} \times 0504 = 6^{\text{m}}\cdot 0.$$

Then, since

$$\left.\begin{matrix}\delta F_2\\ \delta G_2\\ \delta A_2\end{matrix}\right\} = P_, \left.\begin{matrix}F_s\\ G_s\\ A_s\end{matrix}\right\}, \quad \left.\begin{matrix}\delta F_1\\ \delta G_1\\ \delta A_1\end{matrix}\right\} = \cdot 317 P_, \left.\begin{matrix}F_,\\ G_,\\ A_,\end{matrix}\right\},$$

it follows that

$$\delta I_{P_,} = \frac{6^{\text{m}}\cdot 0}{\text{D}} \cos(\odot + 74^\circ)\{F_s \cos\Delta + G_s \sin\Delta \pm \cdot 317\,(F_, \cos\tfrac{1}{2}\Delta + 2G_, \sin\tfrac{1}{2}\Delta)\}$$

$$\delta h_{P_,} = \cdot 050 \cos(\odot + 74^\circ)\{A_s \cos\Delta - F_s \sin\Delta \pm \cdot 317\,(A_, \cos\tfrac{1}{2}\Delta - 2F_, \sin\tfrac{1}{2}\Delta)\}.$$

These must be deemed to be corrections to 𝔦 and 𝔥, since they change signs in six months.

§ 12. *The Formulæ for a Low Water Table*

The formulæ (27), (28), (29), (30), for the height of water would (except in one detail) be applicable to L. W., but they would not be convenient, because Δ oscillates about $-\pi$ at the L. W. which precedes the H. W. for which mean Δ is zero.

Hence put $\Delta = \delta - \pi$.

The L. W. formulæ may be made exactly similar to those for H. W. by making the heights negative, and this condition is satisfied by adding π to all the arguments. Thus the formulæ for the height may be written

$$-\Sigma\, \text{H} \cos(p\,\delta + \theta),$$

where

$$\theta = \vartheta + (1 - p)\,\pi.$$

A similar change may be made in the nodal terms, but the parallactic terms require further consideration. The term $\cos(p_m \Delta + {}_m\vartheta)$ changes, not only because its sign is to be changed and Δ is to be replaced by $\delta - \pi$, but also because, as appears in § 3, ${}_m\vartheta$ changes.

For H. W.

$${}_m\vartheta = p_m\kappa_m - {}_m\kappa = p_n\kappa_m - \kappa_n + \frac{\kappa_m - \kappa_n}{9\cdot 11\text{H}_n/\text{H}_m - 1},$$

but for L. W. κ_m must be replaced by $\kappa_m - \pi$ wherever it is multiplied by p; hence, for L. W., $p_m\kappa_m - {}_m\kappa$ must have $(p_m - p_n)\,\pi$ or $(1 - p_n)\,\pi$ added to its previous value.

Hence the term in $\cos(p_m\Delta + {}_m\vartheta)$ for H. W. corresponds in the expression for $-h_P$ for L. W., to a term involving

$$\cos[p_m\delta + (1 - p_m)\,\pi + (1 - p_n)\,\pi + {}_m\vartheta]$$

Since $p_m = 1$, the required term is $\cos(p_m\delta + {}_m\theta)$, where

$$ {}_m\theta = {}_m\vartheta + (1 - p_{\prime\prime})\,\pi. $$

Again the parallactic term for O involves $\cos({}_o p\Delta + {}_o\vartheta)$ for H. W , and treating it in the same way, the corresponding term in $-h_p$ for L. W. is found to be

$$ \cos({}_o p\delta + {}_o\theta), $$

where

$$ {}_o\theta = (1 - {}_o p)\,\pi + (p_o - p_q)\,\pi + {}_o\vartheta. $$

But

$$ {}_o p = p_o - \frac{\epsilon}{\beta}. $$

Hence

$$ {}_o\theta = {}_o\vartheta + \left(1 + \frac{\epsilon}{\beta} - p_q\right)\pi. $$

We have also

$$ {}_{\prime\prime}\theta = {}_{\prime\prime}\vartheta + (1 - {}_{\prime\prime}p)\,\pi, \qquad {}_{\prime}\theta = {}_{\prime}\vartheta + (1 - {}_{\prime}p)\,\pi. $$

Then the L. W. formulæ for depths below mean sea level are similar in form to those for H. W. for elevation above mean sea level, with δ in place of Δ, and with θ's in the place of ϑ's.

The connection of θ's with ϑ's is given in the following table.

Initial.	Principal and nodal terms.	Parallactic terms.
M_2	$\theta_m = \vartheta_m = 0$	${}_m\theta = {}_m\vartheta + (1 - p_{\prime\prime})\,\pi = {}_m\vartheta + 3^\circ{\cdot}4$
K_2	$\theta_{\prime\prime} = \vartheta_{\prime\prime} + (1 - p_{\prime\prime})\,\pi$ $= \vartheta_{\prime\prime} - 7^\circ{\cdot}0$	${}_{\prime\prime}\theta = \vartheta_{\prime\prime} + (1 - {}_{\prime\prime}p)\,\pi$ $= {}_{\prime\prime}\vartheta - 13^\circ{\cdot}7$
S_2	$\theta_s = \vartheta_s + (1 - p_s)\,\pi$ $= \vartheta_s - 6^\circ{\cdot}4$	(assume $\beta = \frac{4}{3}$)
O	$\theta_o = \vartheta_o + (1 - p_o)\,\pi$ $= \vartheta_o + \frac{1}{2}\pi + 3^\circ{\cdot}4$	${}_o\theta = {}_o\vartheta + (1 + \frac{\epsilon}{\beta} - p_q)\,\pi$ $= {}_o\vartheta + \frac{1}{2}\pi + 8^\circ{\cdot}5$
K_1	$\theta_{\prime} = \vartheta_{\prime} + (1 - p_{\prime})\,\pi$ $= \vartheta_{\prime} + \frac{1}{2}\pi - 3^\circ{\cdot}4$	${}_{\prime}\theta = {}_{\prime}\vartheta + (1 - {}_{\prime}p)\,\pi$ $= {}_{\prime}\vartheta + \frac{1}{2}\pi - 6^\circ{\cdot}8$
P	$\theta_p = \vartheta_p + (1 - p_p)\,\pi$ $= \vartheta_p + \frac{1}{2}\pi - 2^\circ{\cdot}9$	
M_4	$\theta_{2m} = \vartheta_{2m} + (1 - 2p_m)\,\pi$ $= \vartheta_{2m} - \pi$	
S_4	$\theta_{2s} = \vartheta_{2s} + (1 - 2p_s)\,\pi$ $= \vartheta_{2s} - \pi - 12^\circ{\cdot}8$	

All the θ's differ from the ϑ's by a small angle, or by an angle nearly equal to 90° or 180°. Where the difference is small the A, G, F for L. W. will be nearly equal to those for H. W.; where the difference is nearly 90° the A, G, F for L. W. will be nearly the same as those for H. W., a quarter year earlier or later; and where the difference is nearly 180° the A, G, F for L. W. will be nearly equal and opposite to those for H. W. Hence we may set aside the change of 90° and 180° to be satisfied by a shift of a quarter year, or by change of sign. Suppose then, that $\theta = \vartheta + \alpha$, where α is small, and let [A], [G], [F] denote the values of L. W. A, G, F; then, remembering that

$$A = H\cos\vartheta, \qquad G = Hp^2\cos\vartheta, \qquad F = Hp\sin\vartheta,$$

and that [A], [G], [F] are represented by similar formulæ with θ in place of ϑ, we have

$$[A] = A - \frac{\sin\alpha}{p}F,$$
$$[G] = G - p\sin\alpha\, F,$$
$$[F] = F + p\sin\alpha\, A.$$

The values of α are given above, and those of the p's are known; hence it is easy to compute formulæ of transition from L. W. to H. W.

The rules given below in the example are derived from these formulæ, but the coefficients $\sin\alpha/p$ and $p\sin\alpha$ are given in round numbers appropriate for computation, and are sometimes treated as zero.

PART II.—COMPUTATION.

Remarks on the Computations.

The multiplications are supposed to be done with CRELLE'S BREMIKER'S multiplication table.*

The other tables required are tables of squares, natural tangents, circular measure, and a traverse table. BOTTOMLEY'S tables† are convenient for the purpose, because they give no more than is required. The nautical traverse table, such as that in INMAN'S tables, or in CHAMBERS' logarithms, is used for finding such quantities as $H\cos\vartheta$ and $H\sin\vartheta$, for if H is "Distance," $H\cos\vartheta$ is "Lat." and $H\sin\vartheta$ is "Dep.," and the position of the decimal point is determined by inspection. For the use of this table it is advantageous to have only angles with a whole number of degrees, so as to avoid cross interpolations‡; the whole calculation is therefore conducted so as to avoid broken degrees. A traverse table is commonly given for

* 'Rechentafeln,' Berlin, GEORG REIMER.

† 'Four-figure Mathematical Tables,' by J. T. BOTTOMLEY, F.R.S. MACMILLAN.

‡ INMAN'S Traverse Table is arranged so that the interpolation for a fraction of a degree is not very awkward.

"Distances" from 0 to 300, hence, if the "Distance" involves three digits and lies between 300 and 999, an interpolation is required, so as to use the entries between 30 and 99; this interpolation can easily be made by inspection.

All the angles are entered so that the significant part is less than 90°, by treating them as + or −, with π or 180° added where necessary. This facilitates the use of the traverse table.

It is best to determine the signs of the cosines and sines independently from their numerical values, and, accordingly, in the example where ·000 is entered as the value of a sine or cosine, it has a sign attached to it.

I suppose the computer to be able to add up a short column of figures, where some of the entries are + and others −. This is an arithmetical process not much practised, but easily acquired.

The sequences of angles and of cosines and sines, which occur frequently below, appertain (except in the cases of Sa and Ssa) to values of the excess of moon's longitude over sun's (for which the symbol used is ☽) at intervals of 15°, beginning with ☽ = 0°, and ending with ☽ = 345°, 24 values in all. But in the earlier part of the computation, the beginning of the sequence occurs at a different part of the column at different times of the year. Thus, a list of months is written in the margin, to show where we are to begin at any specified time of year. Strictly speaking these months are the times when the sun's longitude + 5° (for which the symbol used is ☉) is equal to a multiple of 30°; thus, when ☉ is 0°, we have March 15th, when ☉ is 30°, April 14th, and so on, as shown in Table VI. If the number of degrees in ☽ be reduced to time, at the rate of 15° per hour, we have, approximately, the time of the moon's transit, and in the later stages of the computation the time of Moon's transit is made to replace ☽. The sequences of angles are found by adding multiples of 30°, adding or subtracting multiples of 15°, or adding multiples of 60° (see Table II.) to certain initial angles (see Table I.). When the sequence has been carried so far that the next addition would reproduce the first angle with π added to it, it is unnecessary to proceed further. In the sequences of cosines and sines of such angles, when we have got to this same point, it is unnecessary to proceed further, since the remainder is the same as the beginning, with the sign changed.

In subsequent stages where a constant has to be added to a sequence, the new sequence will have double as many entries as the old, the first half being formed by addition, and the second half by subtraction; but in repeating the new sequence the signs are *not* to be changed. This follows immediately from what has been said of the signs of sequences.

Before proceeding with the computations I give some tables and rules of general applicability to all ports. It will be best for the computer who is learning the process to pass straight to the example, and to refer back to these tables as they are required; but I give them in the first place, because they will be wanted in the case of any other port.

Tables and Rules applicable to all Ports

TABLE I.—For finding the K's, the initial entries of the several sequences, for H. W. (See (26), § 5.)

Initials.	Principal and nodal terms.	Parallactic terms.
M_2	$K_m = p_m \kappa_m - \kappa_m = 0$	${}_mK = p_n \kappa_m - \kappa_n + \dfrac{\kappa_m - \kappa_n}{9{\cdot}11\, H_n/H_m - 1}$
K_2	$K_{,,} = p_{,,} \kappa_m - \kappa_{,,} - 10°$	${}_{,,}K = K_{,,} + ({}_{,,}p - p_{,,}) \kappa_m = K_{,,} + {\cdot}038\, \kappa_m$
S_2	$K_s = p_s \kappa_m - \kappa_s$	
O	$K_o = p_o \kappa_m - \kappa_o + 95°$	${}_oK = p_q \kappa_m - \kappa_q - \dfrac{{\cdot}013}{\beta} \kappa_m + \dfrac{\kappa_o - \kappa_q}{9{\cdot}11\, H_q/H_o - 1} + 95°$
K_1	$K_{,} = p_{,} \kappa_m - \kappa_{,} - 95°$	${}_{,}K = K_{,} + ({}_{,}p - p_{,}) \kappa_m = K_{,} + {\cdot}0192\, \kappa_m$
P	$K_p = p_p \kappa_m - \kappa_p + 95°$	
M_4	$K_{2m} = 2p_m \kappa_m - \kappa_{2m}$	
S_4	$K_{2s} = 2p_s \kappa_m - \kappa_{2s}$	
Sa	$K_{sa} = - \kappa_{sa} - 5°$	
Ssa	$K_{ssa} = - \kappa_{ssa} - 10°$	

TABLE II.—For finding the sequences of the ϑ's by putting n successively equal to 0, 1, 2, 3, &c. (See (26), § 5.)

Initials.	Principal and nodal terms. ϑ	Parallactic terms. ϑ
M_2	K_m	${}_mK$
K_2	$K_{,,} + 30°n$	${}_{,,}K + 30°n$
S_2	$K_s + 30°n$	
O	$K_o - 15°n$	${}_oK - 15°n$
K_1	$K_{,} + 15°n$	${}_{,}K + 15°n$
P	$K_p + 15°n$	
M_4	K_{2m}	
S_4	$K_{2s} + 60°n$	
Ssa	$K_{ssa} + 60°n$	
Sa	$K_{sa} + 10°n$	

N.B.—The sequence for Sa is required under conditions which differ from those of the other ϑ's.

TABLE III.—The numerical values of the p's. (See § 5.)

Initials.	Principal and nodal terms.	Parallactic terms.
M_2	$p_m = 1$, $p_m{}^2 = 1$	
K_2	$p_{\prime\prime} = 1{\cdot}038$, $p_{\prime\prime}{}^2 = 1{\cdot}078$	$_{\prime\prime}p = 1{\cdot}076$, $_{\prime\prime}p^2 = 1{\cdot}158$
S_2	$p_s = 1{\cdot}035$, $p_s{}^2 = 1{\cdot}071$	
N	$p_n = {\cdot}981$	
O	$p_o = {\cdot}481$, $p_o{}^2 = {\cdot}231$	$_op = {\cdot}481 - {\cdot}013\,\beta^{-1}$, $_op^2 = {\cdot}231 - {\cdot}012\,\beta^{-1}$
K_1	$p_\prime = {\cdot}519$, $p_\prime{}^2 = {\cdot}269$	$_\prime p = {\cdot}538$, $_\prime p^2 = {\cdot}289$
P	$p_p = {\cdot}516$, $p_p{}^2 = {\cdot}266$	
Q	$p_q = {\cdot}462$	

TABLE IV.—For computing corrections for reference to moon's transit, viz., δT, δM, δN, and the sequence Θ. (See (55) § 10.)

Sequence. $(-10° + 30°n)$. Θ			
March	$-10°$	June	$\pi - 10°$
	$+20$		$\pi + 20$
April	$+50$	July	$\pi + 50$
	$+80$		$\pi + 80$
May	$\pi - 70$	August	-70
	$\pi - 40$		-40

Repeat the sequence.

δT $= 0^h{\cdot}172 \sin \Theta$.		δM $= 4^m{\cdot}60 \sin \Theta$.		$-4^m{\cdot}22 \cos \Theta$.		δN $= -4^m{\cdot}22 \cos \Theta - 4^m{\cdot}41$.	
	h.		m.		m.		m.
March	$-{\cdot}030$	March	$-0{\cdot}80$	March	$-4{\cdot}16$	March	$-8{\cdot}57$
	$+{\cdot}059$		$+1{\cdot}57$		$-3{\cdot}97$		$-8{\cdot}38$
April	$+{\cdot}132$	April	$+3{\cdot}52$	April	$-2{\cdot}71$	April	$-7{\cdot}12$
	$+{\cdot}169$		$+4{\cdot}53$		$-0{\cdot}73$		$-5{\cdot}14$
May	$+{\cdot}162$	May	$+4{\cdot}32$	May	$+1{\cdot}45$	May	$-2{\cdot}96$
	$+{\cdot}111$		$+2{\cdot}96$		$+3{\cdot}23$		$-1{\cdot}18$
June	$+{\cdot}030$	June	$+0{\cdot}80$	Rep. and ch.		June	$-0{\cdot}25$
	$-{\cdot}059$		$-1{\cdot}57$				$-0{\cdot}44$
July	$-{\cdot}132$	July	$-3{\cdot}52$			July	$-1{\cdot}70$
	$-{\cdot}169$		$-4{\cdot}53$				$-3{\cdot}68$
August	$-{\cdot}162$	August	$-4{\cdot}32$			August	$-5{\cdot}86$
	$-{\cdot}111$		$-2{\cdot}96$				$-7{\cdot}64$

The sequences for δT, δM, δN are to be repeated without change of sign. To find the succession of values for any month we begin with the entry opposite to that month, read on down to the bottom, and then begin again at the top. For example, δT for July begins with − ·132, and then, after going on down to − ·111, it begins again at the top with − ·030.

TABLE V.—For corrections due to part of the equation of time. (See (63) (64) § 10.)

The following is a table of $-3^{m}\cdot85 \sin(\odot + 74^\circ)$ (which I call *c*), and of the same when the hour is unit of time (which I call *d*).

N.B.—286° is the longitude of sun's perigee + 5°, and 74° is its supplement to 360°.

	c. $-3^{m}\cdot85 \sin(\odot + 74^\circ)$.	*d*. $-0^{h}\cdot064 \sin(\odot + 74^\circ)$.
	m.	h.
March . . .	−3·7	−·062
April	−3·7	−·062
May	−2·8	−·046
June	−1·1	−·018
July	+0·9	+·016
August . . .	+2·7	+·045

TABLE VI.—Dates and Limits of Applicability of the Tide-Tables.

Heading for tide-table.	⊙	Applicability by reference to Sun's long. at Moon's transit. Sun's longitude.	Heading for tide-table.	⊙	Applicability by reference to Sun's long. at Moon's transit. Sun's longitude.
	°	° °		°	° °
March 15	0	from 350 to 0	Sept. 17	180	from 170 to 180
,, 25	10	,, 0 ,, 10	,, 28	190	,, 180 ,, 190
April 4	20	,, 10 ,, 20	Oct. 8	200	,, 190 ,, 200
,, 14	30	,, 20 ,, 30	,, 18	210	,, 200 ,, 210
,, 25	40	,, 30 ,, 40	,, 28	220	,, 210 ,, 220
May 5	50	,, 40 ,, 50	Nov. 7	230	,, 220 ,, 230
,, 15	60	,, 50 ,, 60	,, 17	240	,, 230 ,, 240
,, 26	70	,, 60 ,, 70	,, 27	250	,, 240 ,, 250
June 5	80	,, 70 ,, 80	Dec. 7	260	,, 250 ,, 260
,, 16	90	,, 80 ,, 90	,, 16	270	,, 260 ,, 270
,, 26	100	,, 90 ,, 100	,, 26	280	,, 270 ,, 280
July 7	110	,, 100 ,, 110	Jan. 5	290	,, 280 ,, 290
,, 17	120	,, 110 ,, 120	,, 15	300	,, 290 ,, 300
,, 28	130	,, 120 ,, 130	,, 25	310	,, 300 ,, 310
Aug. 7	140	,, 130 ,, 140	Feb. 4	320	,, 310 ,, 320
,, 17	150	,, 140 ,, 150	,, 13	330	,, 320 ,, 330
,, 28	160	,, 150 ,, 160	,, 23	340	,, 330 ,, 340
Sept. 7	170	,, 160 ,, 170	March 5	350	,, 340 ,, 350

This table gives the days of the year on which ⊙, or Sun's longitude + 5°, is nearly equal to a multiple of 10°. These days are used as headings to the several tide-tables. It is intended that the tables shall be used without an interpolation for

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the time of year, which ought strictly to be made. When the time of a particular moon's transit, with reference to which a tide is to be calculated, falls nearly halfway between any two of the specified days, it becomes uncertain which of the two adjoining tables should be used, and the question can only be decided by reference to the Sun's longitude. A column is therefore given of the limits of applicability of the table.

It would be easy, by means of a table of four columns referring to leap year, to give the Greenwich times at which the Sun's longitude is 0°, 10°, 20°, &c., which would be accurate enough for the present purpose during some twenty-five years.

VII.—*The Choice of a Unit of Length.*

In a calculation of this kind it is advantageous to reduce the number of digits as far as possible, consistently with due accuracy, and it is convenient to omit the decimal point when we deal with heights.

The diurnal tides are so various at different places that no general rule can be made to depend on them.

It is required to express as many of the heights as possible by two digits, and it will be best to take such a unit that much of the work shall be conducted with 70's and 80's, but to allow a margin and not to try to bring them into the 90's.

After consideration I think it is best to take such a unit that $\alpha' H_m$ (see below) shall be expressed by 70 or 80. Since α' is usually about $\frac{4}{57}$, or say $\frac{1}{14}$, then when H_m is given in feet and decimals (or any other unit), we are to multiply the heights by a simple factor lying between $14 \times 70 \div H_m$ and $14 \times 80 \div H_m$.

The rule therefore is:—

Multiply the heights by a factor lying between $1000 \div H_m$ *and* $1200 \div H_m$, *and omit decimals.*

In the example below it would have been best to multiply all the heights by 700. We should then have $H_m = 1098$, $H_s = 488$, &c., and this would have made $\alpha' H_m$ equal to 77. The final step in the calculation would then have been to divide all the heights by 700.

In my example I have not followed this plan, and accordingly the decimal point is retained, and an unnecessary number of digits has been written.

VIII.—*Rules for the Calculation of a L. W. Table.*

It will be more convenient to state these rules as part of the example for the Port of Aden, although they are of course generally applicable to all ports.

EXAMPLE OF FORMATION OF A TIDE-TABLE.

TABLE of Constants for the Port of Aden.

		ft.			ft.
M_2	$H_m =$	1·568	O	$H_o =$	·653
	$\kappa_m =$	229°		$\kappa_o =$	38°
S_2	$H_s =$	·697	K_1	$H_, =$	1·299
	$\kappa_s =$	248°		$\kappa_, =$	36°
K_2	$H_{,,} =$	·201	P	$H_p =$	·388
	$\kappa_{,,} =$	244°		$\kappa_p =$	33°
N	$H_n =$	·427	Q	$H_q =$	·151
	$\kappa_n =$	225°		$\kappa_q =$	42°
L	$H_l =$	·046	Sa	$H_{sa} =$	·390
	$\kappa_l =$	230°		$\kappa_{sa} =$	357°
M_4	$H_{2m} =$	·007	Ssa	$H_{ssa} =$	·095
	$\kappa_{2m} =$	314°		$\kappa_{ssa} =$	126°
S_4	$H_{2s} =$	·006		$\mathfrak{A} =$	3·859
	$\kappa_{2s} =$	271°			

These are the results of four years of observation, and are the constants from which the tide-table is to be computed.

Formation of Sequences (see Tables I., II., III., and § 6).

K_2.	S_2.	O.	K_1.
$p_{,,}\kappa_m =$ 238° $-\kappa_{,,} = -$ 244 $-10° = -$ 10 —— $K_{,,} = -$ 16	$p_s\kappa_m =$ 237° $-\kappa_s = -$ 248 —— $K_s = -$ 11	$p_o\kappa_m =$ 110° $-\kappa_o = -$ 38 $+95° =$ 95 —— 167 $K_o = \pi -$ 13	$p_,\kappa_m =$ 119° $-\kappa_, = -$ 36 $-95° = -$ 95 —— $K_, = -$ 12

P.	M_4.	S_4.
$p_p\kappa_m =$ 118° $-\kappa_p = -$ 33 $+95°$ 95 —— 180 $K_p = \pi +$ 0	$2p_m\kappa_m =$ 458° $-\kappa_{2m} = -$ 314 —— 144 $K_{2m} = \pi -$ 36	$2p_s\kappa_m =$ 474° $-\kappa_{2s} = -$ 271 —— 203 $K_{2s} = \pi +$ 23

Sequences of Angles.

M_2. ϑ_m.	K_2. $(K_{\prime\prime} + 30°n)$ $\vartheta_{\prime\prime}$.	S_2. $(K_s + 30°n)$ ϑ_s.	O. $(K_o - 15°n)$ ϑ_o.	K_1. $(K_{\prime} + 15°n)$ $\vartheta_{\prime}$.	P. $(K_p + 15°n)$ ϑ_p.	M_4. (K_{2m}) ϑ_{2m}.	S_4. $(K_{2s} + 60°n)$ ϑ_{2s}.
	°	°	°	°	°	°	°
0°.	Mar. -16	All months. -11	Mar. $\pi - 13$	Mar. -12	Mar. $\pi + 0$	$\pi - 36$	All months. $\pi + 23$
(a const.)	$+14$	$+19$	$\pi - 28$	$+3$	$\pi + 15$	(const.)	$\pi + 83$
	April $+44$	$+49$	April $\pi - 43$	April $+18$	Feb. $\pi + 30$		-37
	$+74$	$+79$	$\pi - 58$	$+33$	$\pi + 45$		
	May $\pi - 76$	$\pi - 71$	May $\pi - 73$	May $+48$	Jan. $\pi + 60$		
	$\pi - 46$	$\pi - 41$	$\pi - 88$	$+63$	$\pi + 75$		
			June $+77$	June $+78$	Dec. $\pi + 90$		
			$+62$	-87	-75		
			July $+47$	July -72	Nov. -60		
			$+32$	-57	-45		
			Aug. $+17$	Aug. -42	Oct. -30		
			$+2$	-27	-15		

Semi-diurnal: $A = H \cos \vartheta$, $G = Hp^2 \cos \vartheta$, $F = Hp \sin \vartheta$.

M_2.	S_2.			K_2.		
	A_s. $(H_s = \cdot 697.)$	G_s. $(H_s p_s^2 = \cdot 746.)$	F_s. $(H_s p_s = \cdot 721.)$	$A_{\prime\prime}$. $(H_{\prime\prime} = \cdot 201.)$	$G_{\prime\prime}$. $(H_{\prime\prime} p_{\prime\prime}^2 = \cdot 217)$	$F_{\prime\prime}$. $(H_{\prime\prime} p_{\prime\prime} = \cdot 209.)$
$H_m = H_m p_m^2 = 1 \cdot 568$	All months. $+ \cdot 684$	$+ \cdot 732$	$- \cdot 137$	Mar. $+ \cdot 193$	$+ \cdot 209$	$- \cdot 058$
$A_m = G_m = 1 \cdot 568$	$+ \cdot 659$	$+ \cdot 705$	$+ \cdot 234$	$+ \cdot 195$	$+ \cdot 211$	$+ \cdot 051$
$F_m = 0$	$+ \cdot 457$	$+ \cdot 489$	$+ \cdot 544$	April $+ \cdot 145$	$+ \cdot 156$	$+ \cdot 145$
(constants)	$+ \cdot 133$	$+ \cdot 142$	$+ \cdot 708$	$+ \cdot 055$	$+ \cdot 060$	$+ \cdot 201$
	$- \cdot 227$	$- \cdot 243$	$+ \cdot 682$	May $- \cdot 049$	$- \cdot 053$	$+ \cdot 203$
	$- \cdot 526$	$- \cdot 563$	$+ \cdot 473$	$- \cdot 140$	$- \cdot 151$	$+ \cdot 150$

Repeat the sequences for K_2 and S_2 changing the signs.

Add M_2 to S_2.

$A_m + A_s$.	$G_m + G_s$.	$F_m + F_s$.
All months. 2·252	2·300	Same as F_s.
2·227	2·273	
2·025	2·057	
1·701	1·710	
1·341	1·325	
1·042	1·005	
·884	·836	
·909	·863	
1·111	1·079	
1·435	1·426	
1·795	1·811	
2·094	2·131	

Repeat without change.

Then write out the three K_2 sequences, with the months, *in extenso*, each on a separate strip of paper; place the $A_{\prime\prime}$ strip opposite to the $A_m + A_s$ table, so that the month for which the sequence is required falls in the first place; *e.g.*, for March put + ·193, for April put + ·145, for May put − ·049, for June put − ·193, &c., in the first place opposite 2·252 of $A_m + A_s$. Then add the $A_m + A_s$ and $A_{\prime\prime}$ tables together in a different way for each of the six months from March to August.

Proceed with the $G_{\prime\prime}$, $F_{\prime\prime}$ strips in the same way. The next following table is formed in this way.

Semi-diurnal: $A_2 = A_m + A_{\prime\prime} + A_s$, $G_2 = G_m + G_{\prime\prime} + G_s$, $F_2 = F_m + F_{\prime\prime} + F_s$.

March.			April.			&c.
A_2.	G_2.	F_2.	A_2.	G_2.	F_2.	Continue as far as August inclusive.
2·445	2·509	−·195	2·397	2·456	+·008	
2·422	2·484	+·285	2·282	2·333	+·435	
2·170	2·213	+·689	1·976	2·004	+·747	
1·756	1·770	+·909	1·561	1·559	+·858	
1·292	1·272	+·885	1·148	1·116	+·740	
·902	·854	+·623	·847	·794	+·422	
·691	·627	+·195	·739	·680	−·008	
·714	·652	−·285	·854	·803	−·435	
·966	·923	−·689	1·160	1·132	−·747	
1·380	1·366	−·909	1·575	1·577	−·858	
1·844	1·864	−·885	1·988	2·020	−·740	
2·234	2·282	−·623	2·289	2·342	−·422	

Repeat without change.

Diurnal: $A = H \cos \vartheta$, $G = Hp^2 \cos \vartheta$, $F = Hp \sin \vartheta$.

O.				K_1.				P.			
	A_o. (H_o = ·653.)	G_o. ($H_o p_o^2$ = ·151.)	F_o. ($H_o p_o$ = ·314.)		$A_\prime$. ($H_\prime$ = 1·299.)	$G_\prime$. ($H_\prime p_\prime^2$ = ·350.)	$F_\prime$. ($H_\prime p_\prime$ = ·674.)		A_p. (H_p = ·388.)	G_p. ($H_p p_p^2$ = ·103.)	F_p. ($H_p p_p$ = ·200.)
Mar.	−·636	−·147	+·071	Mar.	+1·271	+·342	−·140	Mar.	−·388	−·103	·000
	−·577	−·133	+·148		+1·297	+·350	+·035		−·375	−·100	−·052
April	−·477	−·110	+·214	April	+1·235	+·333	+·208	Feb.	−·336	−·089	−·100
	−·346	−·080	+·266		+1·089	+·294	+·367		−·275	−·073	−·141
May	−·191	−·044	+·300	May	+ ·869	+·234	+·501	Jan.	−·194	−·052	−·173
	−·023	−·005	+·314		+ ·590	+·159	+·601		−·100	−·027	−·193
June	+·147	+·034	+·306	June	+ ·270	+·073	+·659	Dec.	·000	·000	−·200
	+·307	+·071	+·278		− ·068	−·018	+·673		+·100	+·027	−·193
July	+·445	+·103	+·230	July	− ·402	−·108	+·641	Nov.	+·194	+·052	−·173
	+·554	+·128	+·166		− ·708	−·191	+·565		+·275	+·073	−·141
Aug.	+·625	+·144	+·092	Aug.	− ·965	−·260	+·451	Oct.	+·336	+·089	−·100
	+·653	+·151	+·011		−1·157	−·312	+·306		+·375	+·100	−·052

Repeat changing signs.

Add together the O and K_1 sequences as they stand above :—

$O + K_1$.

	$A_o + A_,$.	$G_o + G_,$.	$F_o + F_,$.
March	+ ·635	+ ·195	− ·069
	+ ·720	+ ·217	+ ·183
April	+ ·758	+ ·223	+ ·422
	+ ·743	+ ·214	+ ·633
May	+ ·678	+ ·190	+ ·801
	+ ·567	+ ·154	+ ·915
June	+ ·417	+ ·107	+ ·965
	+ ·239	+ ·053	+ ·951
July	+ ·043	− ·005	+ ·871
	− ·154	− ·063	+ ·731
August	− ·340	− ·116	+ ·543
	− ·504	− ·161	+ ·317

Repeat changing signs.

Write out the P sequences *in extenso* with the months on the margin (24 entries), each on a separate strip of paper; place the A_p strip opposite the $A_o + A_,$ table so that any chosen month in one agrees with that month on the other; add the two tables together, making the first entry in the new sequence that opposite which the chosen month is written. For example the April entry in the A_p sequence (completed) is − ·336, and this added to + ·758, the April entry of $A_o + A_,$, gives + ·422, which is the initial entry in the diurnal sequence for A_1 corresponding to April in the following table. G_p and F_p are operated on in the same way.

The first time the computer does this sort of work he may find it convenient to write out the $O + K_1$ sequences *in extenso*, so as to see exactly how the computation runs, but it will be found with a little practice that this is unnecessary.

Diurnal: $A_1 = A_o + A_, + A_p$, $G_1 = G_o + G_, + G_p$, $F_1 = F_o + F_, + F_p$.

March.			April.			&c.
A_1.	G_1.	F_1.	A_1.	G_1.	F_1.	Continue up to August, inclusive.
+ ·247	+ ·092	− ·069	+ ·422	+ ·134	+ ·522	
+ ·345	+ ·117	+ ·131	+ ·368	+ ·114	+ ·685	
+ ·422	+ ·134	+ ·322	+ ·290	+ ·087	+ ·801	
+ ·468	+ ·141	+ ·492	+ ·192	+ ·054	+ ·863	
+ ·484	+ ·138	+ ·628	+ ·081	+ ·018	+ ·865	
+ ·467	+ ·127	+ ·722	− ·036	− ·020	+ ·810	
+ ·417	+ ·107	+ ·765	− ·151	− ·057	+ ·698	
+ ·339	+ ·080	+ ·758	− ·254	− ·090	+ ·538	
+ ·237	+ ·047	+ ·698	− ·340	− ·116	+ ·343	
+ ·121	+ ·010	+ ·590	− ·404	− ·134	+ ·124	
− ·004	− ·027	+ ·443	− ·441	− ·143	− ·104	
− ·129	− ·061	+ ·265	− ·445	− ·144	− ·324	

Quater-diurnal: $A = H \cos \vartheta$, $G = H (2p)^2 \cos \vartheta$, $F = H (2p) \cos \vartheta$.

M_4.			S_4.		
A_{2m}. ($H_{2m} = \cdot007$.)	G_{2m}. ($4H_m = \cdot028$.)	F_{2m}. ($2H_m = \cdot014$.)	A_{2s}. ($H_{2s} = \cdot006$.)	G_{2s}. ($4H_{2s}p_s^2 = \cdot026$.)	F_{2s}. ($2H_{2s}p_s = \cdot012$.)
$-\cdot006$ const.	$-\cdot023$ const.	$+\cdot008$ const.	All months. $-\cdot006$ $-\cdot001$ $+\cdot005$ Repeat and change.	$-\cdot024$ $-\cdot003$ $+\cdot021$ Repeat and change.	$-\cdot005$ $-\cdot012$ $-\cdot007$ Repeat and change.

Quater-diurnal: $A_4 = A_{2m} + A_{2s}$, $G_4 = G_{2m} + G_{2s}$, $F_4 = F_{2m} + F_{2s}$.

A_4.	G_4.	F_4.
All months. $-\cdot012$ $-\cdot007$ $-\cdot001$ $\cdot000$ $-\cdot005$ $-\cdot011$	$-\cdot047$ $-\cdot026$ $-\cdot002$ $+\cdot001$ $-\cdot020$ $-\cdot044$	$+\cdot003$ $-\cdot004$ $+\cdot001$ $+\cdot013$ $+\cdot020$ $+\cdot015$

Repeat without change of sign.

Semi-annual and Mean Water. (See (56), § 10.)

$$-\kappa_{ssa} = 234^\circ$$
$$-10^\circ = -10$$
$$224$$
$$K_{ssa} = \pi + 44$$

Sequence.

$K_{ssa} + 60°n$. ϑ_{ssa}.	$H_{ssa} \cos \vartheta_{ssa}$. ($H_{ssa} = \cdot095$.)	$B_o = \mathfrak{A} + H_{ssa} \cos \vartheta_{ssa}$. ($\mathfrak{A} = 3\cdot859$.)
March $\pi + 44^\circ$ April -76 May -17	$-\cdot068$ $+\cdot023$ $+\cdot091$ Repeat and change.	March 3·791 April 3·882 May 3·950 June 3·927 July 3·836 Aug. 3·768

Annual.

$$-\kappa_{sa} = -357^\circ$$
$$-5^\circ = -5$$
$$K_{sa} = -2$$

Sequence.

$K_{sa} + 10^\circ n$. ϑ_{sa}.			
March	-2°	June	$+88^\circ$
	$+8$		$\pi - 82$
	$+18$		$\pi - 72$
April	$+28$	July	$\pi - 62$
	$+38$		$\pi - 52$
	$+48$		$\pi - 42$
May	$+58$	Aug.	$\pi - 32$
	$+68$		$\pi - 22$
	$+78$		$\pi - 12$

Annual Tide.

$H_{sa} \cos \vartheta_{sa}$. ($H_{sa} = \cdot 390$.)			
	ft.		ft.
March	$+\cdot 390$	June	$+\cdot 014$
	$+\cdot 386$		$-\cdot 054$
	$+\cdot 371$		$-\cdot 121$
April	$+\cdot 344$	July	$-\cdot 183$
	$+\cdot 307$		$-\cdot 240$
	$+\cdot 261$		$-\cdot 290$
May	$+\cdot 207$	Aug.	$-\cdot 331$
	$+\cdot 146$		$-\cdot 362$
	$+\cdot 081$		$-\cdot 381$

Mean Interval i, and Parallactic Correction to i. (See (31) § 6; and (47), § 8.)

$$\kappa_m = 229^\circ$$
$$\tfrac{1}{30}\kappa_m = 7\cdot 633$$
$$\tfrac{1}{30^2}\cdot\tfrac{21}{20}\kappa_m = \cdot 267$$
$$i = 7^{\rm h}\cdot 900$$

Retaining i in hours, parallactic correction to $i = +0^{\rm m}\cdot 08 \times i$
$= +0^{\rm m}\cdot 63$.

N.B.—$0^{\rm m}\cdot 08$ is an absolute constant.

Parallactic Corrections. (See §§ 3, 8.)

$36{\cdot}4\ \mathrm{H}_n$ is greater than $8\ \mathrm{H}_m$; therefore $\alpha = \frac{4}{3}$, $\alpha'\mathrm{H}_m = {\cdot}07\ \mathrm{H}_m = {\cdot}110$,

$36{\cdot}4\ \mathrm{H}_q$ is greater than $8\ \mathrm{H}_o$; therefore $\beta = \frac{4}{3}$, $\beta'\mathrm{H}_o = {\cdot}07\ \mathrm{H}_o = {\cdot}046$.

[N.B.—When either of these inequalities is *less* instead of greater, put

$$\alpha = 12{\cdot}14\ \mathrm{H}_n/\mathrm{H}_m - \tfrac{4}{3}, \quad \alpha'\mathrm{H}_m = {\cdot}639\ \mathrm{H}_n - {\cdot}07\ \mathrm{H}_m.$$
$$\beta = 12{\cdot}14\ \mathrm{H}_q/\mathrm{H}_o - \tfrac{4}{3}, \quad \beta'\mathrm{H}_o = {\cdot}639\ \mathrm{H}_q - {\cdot}07\ \mathrm{H}_o.$$

If the N tide is unknown take $\alpha = 1$, $\alpha' = \frac{1}{19}$; if the Q tide is unknown take $\beta = 1$, $\beta' = \frac{1}{19}$.]

(Table III.) $_op = {\cdot}481 - {\cdot}013\beta^{-1} = {\cdot}481 - {\cdot}010 = {\cdot}471$;

$$_op^2 = {\cdot}231 - {\cdot}012\beta^{-1} = {\cdot}231 - {\cdot}009 = {\cdot}222.$$

(Table I.) Let

$$\gamma = \frac{\kappa_m - \kappa_n}{9{\cdot}11\ \mathrm{H}_n/\mathrm{H}_m - 1};$$

the denominator is $9{\cdot}11 \times {\cdot}272 - 1 = 1{\cdot}48$; the numerator is $4°$. Hence

$$\gamma = +4° \div 1{\cdot}48 = +3°.$$

Let

$$\delta = \frac{\kappa_o - \kappa_q}{9{\cdot}11\ \mathrm{H}_q/\mathrm{H}_o - 1} - \frac{{\cdot}013}{\beta}\kappa_m;$$

the denominator of the first term is $9{\cdot}11 \times {\cdot}232 - 1 = 1{\cdot}11$; the numerator is $-4°$. Hence the first term is $-4°$. The second term is $-{\cdot}010 \times 229° = -2°$. Hence

$$\delta = -6°.$$

(See Tables I., II., III.)

$_2$.	K$_2$.	O.	K$_o$.
°	°	°	°
$p_n\kappa_m = 225$	${\cdot}0384\kappa_m = 9$	$p_q\kappa_m = 106$	${\cdot}0192\kappa_m = 4$
$-\kappa_n = -225$	$\mathrm{K}_{\prime\prime} = -16$	$-\kappa_q = -42$	$\mathrm{K}_{\prime} = -12$
$+\gamma\ \ 3$		$+\delta = -6$	
		$+95° = 95$	
		153	
$_mK = +3$	$_{\prime\prime}K = -7$	$_oK = \pi - 27$	$_{\prime}K = -8$

N.B.—My calculations were made on a principle, now abandoned, which led to slightly different values. I therefore now continue the calculation with

$$\alpha'\mathrm{H}_m = {\cdot}122, \quad \beta'\mathrm{H}_o = {\cdot}041,$$
$$_m\mathrm{K} = +5°, \quad _{\prime\prime}\mathrm{K} = -8°, \quad _o\mathrm{K} = \pi - 19°, \quad _{\prime}\mathrm{K} = -8°.$$

Sequences of Angles.

M_2. (${}_m$K.) ${}_m\vartheta$.		K_2. (${}_{,,}$K + 30° n) ${}_{,,}\vartheta$.		O. (${}_o$K − 15° n) ${}_o\vartheta$.		K_1. (${}_{,}$K + 15° n) ${}_{,}\vartheta$.
°		°		°		°
+ 5	March	− 8	March	π − 19	March	− 8
const.		+ 22		π − 34		+ 7
	April	+ 52	April	π − 49	April	+ 22
		+ 82		π − 64		+ 37
	May	π − 68	May	π − 79	May	+ 52
		π − 38		+ 86		+ 67
			June	+ 71	June	+ 82
				+ 56		π − 83
			July	+ 41	July	π − 68
				+ 26		π − 53
			Aug.	+ 11	Aug.	π − 38
				− 4		π − 23

Semi-diurnal.

$${}_mA = \alpha' H_m \cos {}_m\vartheta, \quad {}_mG = \alpha' H_m p_m^2 \cos {}_m\vartheta, \quad {}_mF = \alpha' H_m p_m \sin {}_m\vartheta;$$

$${}_{,,}A = \cdot 036\ H_{,,} \cos {}_{,,}\vartheta, \quad {}_{,,}G = \cdot 036\ H_{,,}\ {}_{,,}p^2 \cos {}_{,,}\vartheta, \quad {}_{,,}F = \cdot 036\ H_{,,}\ {}_{,,}p \sin {}_{,,}\vartheta.$$

$$z_2 = {}_mA + {}_{,,}A, \quad m_2 = {}_mG + {}_{,,}G, \quad l_2 = {}_mF + {}_{,,}F.$$

${}_m$A. ($\alpha' H_m$ = ·122.)		${}_{,,}$A. (·036 $H_{,,}$ = ·007.)	${}_m$G. ($\alpha' H_m p_m^2$ = ·122.)		${}_{,,}$G. (·036 $H_{,,}\ {}_{,,}p^2$ = ·008.)	${}_m$F. ($\alpha' H_m p_m$ = ·122.)		${}_{,,}$F. (·036 $H_{,,}\ {}_{,,}p$ = ·007.)
+ ·122	March	+ ·007	+ ·122	March	+ ·008	+ ·011	March	− ·001
const.		+ ·006	const.		+ ·007	const.		+ ·003
	April	+ ·004		April	+ ·005		April	+ ·006
		+ ·001			+ ·001			+ ·007
	May	− ·003		May	− ·003		May	+ ·006
		− ·006			− ·006			+ ·004

	z_2		m_2		l_2
March	+ ·129	March	+ ·130	March	+ ·010
	+ ·128		+ ·129		+ ·014
April	+ ·126	April	+ ·127	April	+ ·017
	+ ·123		+ ·123		+ ·018
May	+ ·119	May	+ ·119	May	+ ·017
	+ ·116		+ ·116		+ ·015
June	+ ·115	June	+ ·114	June	+ ·012
	+ ·116		+ ·115		+ ·008
July	+ ·118	July	+ ·117	July	+ ·005
	+ ·121		+ ·121		+ ·004
Aug.	+ ·125	Aug.	+ ·125	Aug.	+ ·005
	+ ·128		+ ·128		+ ·007

Repeat without change.

It might suffice if the parallactic correction to K_2 were neglected, in which case $z_2 = m_2 = {}_mA$, $l_2 = {}_mF$. The labour of making the correct table is, however, inconsiderable.

Diurnal: ${}_oA = \beta' H_o \cos {}_o\vartheta$, ${}_oG = \beta' H_o\, {}_op^2 \cos {}_o\vartheta$, ${}_oF = \beta' H_o\, {}_op \sin {}_o\vartheta$,

${}_,A = \cdot036 H_, \cos {}_,\vartheta$, ${}_,G = \cdot036 H_,\, {}_,p^2 \cos {}_,\vartheta$, ${}_,F = \cdot036 H_,\, {}_,p \sin {}_,\vartheta$,

$z_1 = {}_oA + {}_,A$, $m_1 = {}_oG + {}_,G$, $l_1 = {}_oF + {}_,F$.

	${}_oA$. ($\beta' H_o = \cdot041$.)	${}_,A$. ($\cdot036 H_, = \cdot047$.)	${}_oG$. ($\beta' H_o\, {}_op^2 = \cdot009$.)	${}_,G$. ($\cdot036 H_,\, {}_,p^2 = \cdot013$.)	${}_oF$. ($\beta' H_o\, {}_op = \cdot019$.)	${}_,F$. ($\cdot036 H_,\, {}_,p = \cdot025$.)
March	−·039	+·047	−·009	+·013	+·006	−·003
	−·034	+·047	−·007	+·013	+·011	+·003
April	−·027	+·044	−·006	+·012	+·014	+·009
	−·018	+·038	−·004	+·010	+·017	+·015
May	−·008	+·029	−·002	+·008	+·019	+·020
	+·003	+·018	+·001	+·005	+·019	+·023
June	+·013	+·007	+·003	+·002	+·018	+·025
	+·023	−·006	+·005	−·002	+·016	+·025
July	+·031	−·018	+·007	−·005	+·012	+·023
	+·037	−·028	+·008	−·008	+·008	+·020
Aug.	+·040	−·037	+·009	−·010	+·004	+·015
	+·041	−·043	+·009	−·012	−·001	+·010

	z_1		m_1		l_1
March	+·008	March	+·004	March	+·003
	+·013		+·006		+·014
April	+·017	April	+·006	Apr.	+·023
	+·020		+·006		+·032
May	+·021	May	+·006	May	+·039
	+·021		+·006		+·042
June	+·020	June	+·005	June	+·043
	+·017		+·003		+·041
July	+·013	July	+·002	July	+·035
	+·009		·000		+·028
Aug.	+·003	Aug.	−·001	Aug.	+·019
	−·002		−·003		+·009

Repeat these sequences, changing the signs.

Nodal Corrections. (See (51) (52) § 9.)

Find by reference to preceding sequences the following nine sequences:—

(i.) $- \cdot0372\, A_m$, (ii.) $+ \cdot283\, A_{,,}$, (iii.) $+ \cdot296\, F_{,,}$, (iv.) $+ \cdot283\, F_{,,}$, (v.) $- \cdot0372\, G_m$,
(vi.) $- \cdot296\, G_{,,}$, (vii.) $- \cdot0372\, G_m$, (viii.) $+ \cdot283\, G_{,,}$, (ix.) $+ \cdot319\, F_{,,}$.

Then the semi-diurnal sequences are as follows:—

a_2 is (i.) + (ii.); b_2 is (iii.); c_2 is (iv.); d_2 is (v.) + (vi.); e_2 is (vii.) + (viii.); f_2 is (ix.).

For example:—

	a_2.		a_2.
March . . .	− ·003	June . . .	− ·113
	− ·003		− ·113
April . . .	− ·017	July	− ·099
	− ·042		− ·074
May . . .	− ·072	August . . .	− ·044
	− ·098		− ·018

This, and the other semi-diurnal sequences are repeated without change of sign, and in all six of them the months run just as in this example, and denote the places at which to begin reading the sequence for the month in question.

The diurnal sequences are obtained thus :—

Find, from preceding sequences, the twelve following—

(i.) + ·188 A_o, (ii.) + ·115 $A_{,}$, (iii.) − ·391 F_o, (iv.) + ·297 $F_{,}$, (v.) + ·188 F_o, (vi.) + ·115 $F_{,}$, (vii.) + ·391 G_o, (viii.) − ·297 $G_{,}$, (ix.) + ·188 G_o, (x.) + ·115 $G_{,}$, (xi.) − ·0905 F_o, (xii.) + ·080 $F_{,}$.

Then

a_1 is (i.) + (ii.) ; b_1 is (iii.) + (iv.) ; c_1 is (v.) + (vi.) ;
d_1 is (vii.) + (viii.) ; e_1 is (ix.) + (x.) ; f_1 is (xi.) + (xii.).

For example :—

	a_1.		a_1.
March . . .	+ ·026	June . . .	+ ·059
	+ ·041		+ ·050
April . . .	+ ·052	July	+ ·038
	+ ·060		+ ·023
May	+ ·064	August . . .	+ ·007
	+ ·064		− ·010

This, and the other diurnal sequences, are repeated with change of sign, and the months in all six of them run just as in this example, and denote the places at which to begin reading the sequence for the month in question.

Calculation of Height, Interval, and Corrections for each Month. (See §§ 7, 8, 9, 10.)

Remarks.—Each column in the following computation is arranged exactly like the first, so that it is unnecessary to repeat the letters in the successive columns.

For the month of March, which serves as an example, we refer to the March sequences, and enter the twelve values of G_2 successively, in the top left-hand corners of twelve columns; below these are entered the twelve values of G_1, and the twelve values of G_4, and on the right of the columns are put the twelve values of F_2, F_1, F_4. A similar statement is true of all the other symbols all the way down, and all the sequences are utilised up to twelve entries in each.

The divisions and multiplications may be done by CRELLE's table; Δ_o is found by a table of natural tangents, and $\delta\Delta$ is converted into degrees by a table of circular measure or radians. It is necessary to take as an approximate value of Δ_o the nearest even number of degrees.

From the places where the values of Δ_o are found, the left-hand side of each column corresponds to the time of moon's transit written at the head of the column, and the right-hand side to a time of moon's transit 12^h greater than the time specified. But the whole table for any month serves for its opposite (*e.g.*, September opposite to March), by transposing the words right and left in the preceding statement. Thus the whole computation has only to be made for six months (up to August inclusive), instead of for twelve. The diurnal terms with suffix 1, are written in the margin, with alternative signs, and the upper sign is to be used on the left, and the lower on the right of each column.

Thus, in finding, for example, $F_1 \cos \frac{1}{2}\Delta_o$ on the right we deem the F_1 written at the head to have its sign changed. Thus, in the column of 0^h we have $F_1 = -\cdot069$ and on the right hand, $\Delta_o = +4°$; then the required entry, on the right-hand, for $F_1 \cos \frac{1}{2}\Delta_o$ is $+\cdot069 \cos(+2°) = +\cdot069$.

The values of δT, δM, δN are extracted from the sequences of those functions in Table IV., and they are the same on each side of the column. The value of B_o is taken from the sequence of the semi-annual tide and mean water, and changes only with the month.

The parallactic correction to the mean interval, i, is introduced in computing R. This is a constant of the port and is the same in all months.

In computing the height $\mathfrak{H}$, and its corrections, an approximate value of Δ is used, namely, the nearest even number of degrees; this approximate Δ will often be the same as Δ_o.

In this table it appears to me specially important that the signs of the sines and cosines should be determined independently of their numerical values.

Whereas in the right-hand of column 6^h we get, as a result of a second approximation, no value of Δ, the conjectural value $\Delta = +90°$ is adopted for the computation of P, Q, S, and values of M, N, R are not computed.

The table has rows marked δI, δh, M″, N″, P″, Q″; all these are derived from a subsequent table of "*Corrections for reference to the Moon's transit.*" But it appears convenient to finish off the computation on this sheet, although we have to pause in the computation in order to calculate the said table of corrections.

MARCH.

Interval.	0° or 0h.			15° or 1h.			90° or 6h.	
G_2	+2·509	F_2	− ·195	+2·484	+ ·285		+ ·627	+ ·195
G_1	+ ·092	F_1	− ·069	+ ·117	+ ·131		+ ·107	+ ·765
G_4	− ·047	F_4	+ ·003	− ·026	− ·004	&c.	− ·047	+ ·003
$G_2 + G_1$	+2·601	$F_2 + F_1$	− ·264	+2·601	+ ·416		+ ·734	+ ·960
$G_2 - G_1$	+2·417	$F_2 - F_1$	− ·126	+2·367	+ ·154		+ ·520	− ·570
$\tan \Delta_o = -\frac{F_2 + F_1}{G_2 + G_1}$	+ ·102	$-\frac{F_2 - F_1}{G_2 - G_1}$	+ ·052	− ·160	− ·065		−1·308	+1·096
Δ_o	+ 6°		+ 4°	−10°	−4°		−52°	+48°
a $F_2 \cos \Delta_o$	− ·194		− ·195	+ ·281	+ ·284		+ ·120	+ ·131
β $F_2 \sin \Delta_o$	− ·020		− ·014	− ·050	− ·020		− ·154	+ ·145
γ $G_2 \cos \Delta_o$	+2·495		+2·503	+2·446	+2·478		+ ·386	+ ·420
δ $G_2 \sin \Delta_o$	+ ·262		+ ·175	− ·431	− ·173		− ·493	+ ·466
ϵ $\pm F_1 \cos \frac{1}{2} \Delta_o$	− ·069		+ ·069	+ ·131	− ·131		+ ·688	− ·699
ζ $\pm F_1 \sin \frac{1}{2} \Delta_o$	− ·004		+ ·002	− ·011	+ ·005	&c.	− ·336	− ·311
η $\pm 2G_1 \cos \frac{1}{2} \Delta_o$	+ ·184		− ·184	+ ·233	− ·234		+ ·192	− ·196
θ $\pm 2G_1 \sin \frac{1}{2} \Delta_o$	+ ·010		− ·006	− ·020	+ ·008		− ·094	− ·087
λ $F_4 \cos 2\Delta_o$	+ ·003		+ ·003	− ·004	− ·004		− ·001	·000
μ $F_4 \sin 2\Delta_o$	+ ·001		+ ·000	+ ·001	+ ·001		− ·003	+ ·003
ν $\frac{1}{2}G_4 \cos 2\Delta_o$	− ·023		− ·023	− ·012	− ·013		+ ·006	+ ·002
ρ $\frac{1}{2}G_4 \sin 2\Delta_o$	− ·005		− ·003	+ ·004	+ ·002		+ ·023	− ·023
$a + \delta$	+ ·068		− ·020	− ·150	+ ·111		− ·373	+ ·597
$\epsilon + \theta$	− ·059		+ ·063	+ ·111	− ·123		+ ·594	− ·786
$\lambda + \rho$	− ·002		·000	·000	− ·002		+ ·022	− ·023
Sum N_o	+ ·007		+ ·043	− ·039	− ·014		+ ·243	− ·212
$\zeta - \eta$	− ·188		+ ·186	− ·244	+ ·239		− ·528	− ·115
$\frac{1}{2}(\zeta - \eta)$	− ·094		+ ·093	− ·122	+ ·120		− ·264	− ·058
$\beta - \gamma$	−2·515		−2·517	−2·496	−2·498	&c.	− ·540	− ·275
$2(\mu - \nu)$	+ ·048		+ ·046	+ ·026	+ ·028		− ·018	+ ·002
Sum D_o	−2·561		−2·378	−2·592	−2·350		− ·822	− ·331
$\delta\Delta = N_o/D_o$	− ·003		− ·018	+ ·015	+ ·006		− ·296	+ ·640
(In degrees) $\delta\Delta$	− 0°·2		− 1°·0	+0°·9	+0°·3		−17 ·0	+36°·7
$\Delta = \Delta_o + \delta\Delta$	+5·8		+3·0	−9·1	−3·7		−68·6*	No H. W.*
$\frac{1}{30}\Delta$	·193		·100	·303	·123		2·287	
$\frac{1}{30^2}\frac{21}{20}\Delta$	7		4	11	4		80	
(In hours) I	+ ·200		+ ·104	− ·314	− ·127	&c.	−2·367	
δT	− ·030		− ·030	+ ·059	+ ·059		+ ·030	
	+ ·170		+ ·074	− ·255	− ·068		−2·337	
(Mean int.) i	7·900		7·900	7·900	7·900		7·900	
	8·070		7·974	7·645	7·832		5·563	
	8h 4m·2		7h 58m·4	7h 38m·7	7h 49m·9		5h 33m·8	
(See below) δI	+0·9		+ 0·4	− 1·8	− 0·8		+ 0·6	
$\mathfrak{I}$	8h 5m		7h 59m	7h 37m	7h 49m		5h 34m	

Take upper sign of diurnal terms on left, lower on right.

* Second approximation (see below) introduced here.

Continue up to 165° or 11h, twelve columns in all.

MARCH (continued).

Correction of D.	0° or 0^h.			15° or 1^h.			90° or 6^h.		
$a + \delta$							− ·531*		
$\frac{1}{4}(\epsilon + \theta)$							+ ·126*		
$4(\lambda + \rho)$							+ ·044*		
Sum	This correction need only be made when D_o is small and $\delta\Delta$ considerable						− ·361*		
(In circ. meas.) $\delta\Delta$						&c.	+ ·022*		&c.
Product δD							− ·008*		
D_o							− ·744*		
$D = D_o + \delta D$	−2·56		−2·38	−2·59	−2·35		− ·752*		
Height.									
A_2	2·445	$-F_2$	+ ·195	2·422	− ·285	&c.	·691	− ·195	
A_1	+ ·247	$-F_1$	+ ·069	+ ·345	− ·131		+ ·417	− ·765	
A_4	− ·012	$-F_4$	− ·003	− ·007	+ ·004		− ·012	− ·003	
Approx. Δ	+ 6°		+ 4°	− 10°	− 4°		− 70*	None	
$A_2 \cos\Delta$	2·432	Upper sign on left; lower on right.	2·439	2·385	2·416		·236	See below for evanescent tide.	
$-F_2 \sin\Delta$	+ ·020		+ ·014	+ ·050	+ ·020		+ ·183		
$\pm A_1 \cos\frac{1}{2}\Delta$	+ ·247		− ·247	+ ·344	− ·345		+ ·342		
$\mp 2F_1 \sin\frac{1}{2}\Delta$	+ ·007		− ·005	+ ·023	− ·009		+ ·878		
$A_4 \cos 2\Delta$	− ·012		− ·012	− ·007	− ·007	&c.	+ ·009		&c.
$-\frac{1}{2}F_4 \sin 2\Delta$	− ·000		− ·000	− ·001	− ·000		+ ·001		
Sum h	2·694		2·189	2·794	2·075		1·649		
B_o	3·791		3·791	3·791	3·791		3·791		
	6·485		5·980	6·585	5·866		5·440		
(See below) δh	− ·004		+ ·001	+ ·004	− ·012		− ·014		
𝔥	6·48		5·98	6·59	5·85		5·43		
Nodal correction.									
Δ	+ 6°		+ 4°	− 10°	− 4°		− 70°*	+ 90*	
$119^m \div D$	− 46^m		− 50^m	− 46^m	− 51^m	&c.	− 158^m*		
c_2	− ·016	e_2	+ ·001	+ ·014	+ ·002		+ ·016	− ·117	
c_1	− ·003	e_1	+ ·011	+ ·032	+ ·015		+ ·134	+ ·014	
$c_2 \cos\Delta$	− ·016	Upper sign on left; lower on right.	− ·016	+ ·014	+ ·014		+ ·005		
$e_2 \sin\Delta$	+ 0		+ 0	− 0	− 0		+ ·110		
$\pm c_1 \cos\frac{1}{2}\Delta$	− 3		+ 3	+ 32	− 32		+ ·110		
$\pm 2e_1 \sin\frac{1}{2}\Delta$	+ 1		− 1	− 3	+ 1		− · 16		
Sum	− ·018		− ·014	+ ·043	− ·017	&c.	+ ·209		&c.
Mult. by 119/D	+ 0·8		+ 0·7	− 2·0	+ 0·9		− 33·0		
δM	− 0·8		− 0·8	+ 1·6	+ 1·6		+ 0·8		
Sum M′	0·0		− 0·1	− 0·4	+ 2·5		− 32·2		
(See below) M″	+ 0·4		+ 0·2	− 0·8	− 0·4		+ 0·3		
Sum M	+ 0·4		+ 0·1	− 1·2	+ 2·1		− 31·9	None	

MARCH (continued).

Nodal correction—(continued).	0° or 0^h.			15° or 1^h.			90° or 6^h.		
d_2	— ·120	f_2	— ·019	&c.	&c.	&c.	&c.	&c.	&c.
d_1	— ·159	f_1	— ·017						
$d_2 \cos \Delta$									
$f_2 \sin \Delta$	&c.								
$\pm d_1 \cos \frac{1}{2}\Delta$									
$\pm 2f_1 \sin \frac{1}{2}\Delta$		Compute like M		&c.	&c.	&c.	&c.	&c.	&c.
Sum									
Mult. by 119/D	&c.								
δN									
Sum N′	&c.								
(See below) N″									
Sum N								None	
a_2	— ·003	$-c_2$	+ ·016	— ·003	— ·014		— ·113	— ·016	
a_1	+ ·026	$-c_1$	+ ·003	+ ·041	— ·032	&c.	+ ·059	— ·134	
$a_2 \cos \Delta$	— ·003	Upper sign on left; lower on right.	— ·003	— ·003	— ·003		— ·039	·000	
$-c_2 \sin \Delta$	+ 2		+ 1	+ 2	+ 1		+ 15	— 16	
$\pm a_1 \cos \frac{1}{2}\Delta$	+ 26		— 26	+ 41	— 41		+ 48	— 42	
$\mp 2c_1 \sin \frac{1}{2}\Delta$	+ 0		— 0	+ 6	— 2		+ ·154	+ ·190	&c.
Sum P′	+ ·025		— ·028	+ ·046	— ·045		+ ·178	+ ·132	
(See below) P″	— ·002		·000	+ ·002	— ·005		— ·006	+ ·005	
Sum P	+ ·023		— ·028	+ ·048	— ·050		+ ·172	+ ·137	
b_2	— ·017	$-d_2$	+ ·120	&c.	&c.		&c.	&c.	
b_1	— ·070	$-d_1$	+ ·159						
$b_2 \cos \Delta$									
$-d_2 \sin \Delta$	&c.								
$\pm b_1 \cos \frac{1}{2}\Delta$									
$\mp 2d_1 \sin \frac{1}{2}\Delta$		Compute like P		&c.	&c.	&c.	&c.	&c.	&c.
Sum Q′	&c.								
(See below) Q″									
Sum Q									
Parallactic corrections.									
l_2	+ ·010	m_2	+ ·130	+ ·014	+ ·129	&c.	+ ·012	+ ·114	&c.
l_1	+ ·003	m_1	+ ·004	+ ·014	+ ·006		+ ·043	+ ·005	
$l_2 \cos \Delta$	+ ·010	Upper sign on left; lower sign on right.	+ ·010	+ ·014	+ ·014		+ ·004		
$m_2 \sin \Delta$	+ 14		+ 9	— 22	— 9		— ·107		
$\pm l_1 \cos \frac{1}{2}\Delta$	+ 3		— 3	+ 14	— 14		+ 35		
$\pm 2m_1 \sin \frac{1}{2}\Delta$	+ 0		— 0	— 1	+ 0	&c.	— 6		&c.
Sum	+ ·027		+ ·016	+ ·005	— ·009		— ·074		
Mult. by 119/D	— 1·2		— 0·8	— 0·2	+ 0·5		+ 11·7		
Par. corr. to i	+ 0·6		+ 0·6	+ 0·6	+ 0·6		+ 0·6		
Sum R	— 0·6		— 0·2	+ 0·4	+ 1·1		+ 12·3	None	

MARCH (continued)

Parallactic corrections —(continued).	0° or 0ʰ.			15° or 1ʰ.			30° or 6ʰ.		
z_2 z_1	+ ·129 + ·008	$-l_2$ $-l_1$	− ·010 − ·003	&c.	&c.	&c.	&c.	&c.	&c.
$z_2 \cos \Delta$ $-l_2 \sin \Delta$ $\pm z_1 \cos \frac{1}{2}\Delta$ $\mp 2l_1 \sin \frac{1}{2}\Delta$	&c. &c.	Compute like		P and Q		&c.	&c.	&c.	&c.
Sum S	+ ·135		+ ·120	+ ·143	+ ·115		+ ·115	+ ·035	

SECOND approximation.

When the correction $\delta\Delta$ is large, as in the case of column 6ʰ, this is necessary.

Column of 90° or 6ʰ.		
Assume Δ_o	−70°	+86°
α	+·067	+·014
β	−·183	+·195
γ	+·214	+·044
δ	−·598	+·625
ϵ	+·627	−·560
ζ	−·439	−·529
η	+·175	−·157
θ	−·123	−·146
λ	−·002	−·003
μ	−·002	+·000
ν	+·018	+·023
	+·015	−·003
$\alpha + \delta$	−·531	+·639
$\epsilon + \theta$	+·504	−·706
$\lambda + \rho$	+·011	−·006
Sum N_o	−·016	−·073
$\zeta - \eta$	−·614	−·377
$\frac{1}{2}(\zeta - \eta)$	−·307	−·189
$\beta - \gamma$	−·397	+·151
$2(\mu - \nu)$	−·040	−·046
Sum D_o	−·744	−·084
$\delta\Delta = N_o/D_o$	+·022	+·87
(In degrees) $\delta\Delta$	+1°·4	+50°
$\Delta = \Delta_o + \delta\Delta$	−68·6	No H. W.

It is concluded that as the correction in the second column is 50° there is no H. W. A conjectural value of $\Delta = +90°$ is used above in computing P, Q, S.*

* There ought in strictness to be further corrections to 𝔍 and 𝔥, but they are of little importance Thus:—

Further correction to 𝔍 and 𝔥.

When Δ is greater than (say) 50° there are further corrections $[\delta I]$, $[\delta h]$ computed from

$$[\delta I] = -2^{\mathrm{h}}(1 - \tfrac{1}{85})\frac{\cdot 012}{\mathrm{D}}[3\,\mathrm{F}_2 \sin^2\Delta\cos\Delta + (\mathrm{G}_{,,} + \mathrm{G}_s)\sin^3\Delta]$$

$$[\delta h] = -\cdot 012\,[3\,(\mathrm{A}_{,,} + \mathrm{A}_s)\sin^2\Delta\cos\Delta - \mathrm{F}_2\sin^3\Delta]$$

Thus in the column of 6^{h} on the left $\Delta = -70°$; then compute thus:—

$\mathrm{F}_2 = +\cdot 195$, $\quad 3\mathrm{F}_2 = +\cdot 585$,

$\mathrm{G}_2 = +\ \cdot 627$, $\quad -\mathrm{H}_m = -1\cdot 568$, $\quad \mathrm{G}_2 - \mathrm{H}_m = -\ \cdot 941$, $\quad \mathrm{G}_{,,} + \mathrm{G}_s = -\ \cdot 941$

$\mathrm{A}_2 = +\ \cdot 691$, $\quad -\mathrm{H}_m = -1\cdot 568$, $\quad \mathrm{A}_2 - \mathrm{H}_m = -\ \cdot 877$, $\quad 3\,(\mathrm{A}_2 - \mathrm{H}_m) = -2\cdot 631$, $\quad 3\,(\mathrm{A}_{,,} + \mathrm{A}_s) = -2\cdot 631$

By successive use of Traverse table,—

$(\mathrm{G}_{,,} + \mathrm{G}_s)\sin\Delta = +\cdot 884$, $\quad 3\,\mathrm{F}_2\sin\Delta = -\cdot 550$

$(\mathrm{G}_{,,} + \mathrm{G}_s)\sin^2\Delta = -\cdot 831$, $\quad 3\,\mathrm{F}_2\sin^2\Delta = +\cdot 517$

$(\mathrm{G}_{,,} + \mathrm{G}_s)\sin^3\Delta = +\ \cdot 781$

$+\,3\,\mathrm{F}_2\sin^2\Delta\cos\Delta = +\ \cdot 177$

$+\ \cdot 958$

$\times\ -\ \cdot 012$

Divide by D or $-\cdot 752$ $\quad -\cdot 0115$

$\times\ 2^{\mathrm{h}}$ $\quad +\cdot 0153$

$+\cdot 0306$

$-\frac{1}{85}$ $\quad -\ 4$

$[\delta I] = +\ \cdot 030$

$= +\ 1^{\mathrm{m}}\cdot 8$

Previous 𝔍 = $5^{\mathrm{h}}\ 33\cdot 8$

Correct 𝔍 = $5^{\mathrm{h}}\ 36^{\mathrm{m}}$

Again, by successive use of Traverse table,—

$3\,(\mathrm{A}_{,,} + \mathrm{A}_s)\sin\Delta = +2\cdot 472$, $\quad -\mathrm{F}_2\sin\Delta = +\cdot 183$

$3\,(\mathrm{A}_{,,} + \mathrm{A}_s)\sin^2\Delta = -2\cdot 323$, $\quad -\mathrm{F}_2\sin^2\Delta = -\cdot 172$

$3\,(\mathrm{A}_{,,} + \mathrm{A}_s)\sin^2\Delta\cos\Delta = -\ \cdot 795$

$-\mathrm{F}_2\sin^3\Delta = +\ \cdot 162$

$-\ \cdot 633$

$\times\ -\cdot 012$

$[\delta h] = +\ \cdot 008$

Previous 𝔥 = $5\cdot 426$

Correct 𝔥 = $5\cdot 43$

Evanescent Tide. (See § 7.)

The right-hand column of 6^h leads to no H. W., and the tables of $\mathfrak{J}$ and $\mathfrak{H}$ must be completed by other formulæ.

The following calculation is very like the preceding one. The value Δ_o will always be nearly $\pm$ 90°, and in our example it is exactly + 90°.

The computation of M, N, R is to be omitted, and that of P, Q, S has been included in the general calculation with a conjectural $\Delta = + 90°$.

MARCH.

Evanescent Tide.			
	90° or 6^h.		
G_2	+ ·627	F_2	+ ·195
G_1	+ ·107	F_1	+ ·765
G_4	− ·047	F_4	+ ·003
		$\frac{1}{4} F_1$	+ ·191
$G_2 + G_1$	,,	$F_2 + \frac{1}{4} F_1$	,,
$G_2 - G_1$	+ ·520	$F_2 - \frac{1}{4} F_1$	+ ·004
$\tan \Delta_o = \frac{G_2 + G_1}{F_2 + \frac{1}{4} F_1}$	,,	$\frac{G_2 - G_1}{F_2 - \frac{1}{4} F_1}$	+ 130·0
Δ_o	,,		+ 90°
α $F_2 \cos \Delta_o$	,,		·000
β $F_2 \sin \Delta_o$	,,		+ ·195
γ $G_2 \cos \Delta_o$	,,		·000
δ $G_2 \sin \Delta_o$	,,		+ ·627
$\epsilon \pm F_1 \cos \frac{1}{2} \Delta_o$	,,		− ·541
$\zeta \pm F_1 \sin \frac{1}{2} \Delta_o$	,,		− ·541
$\eta \pm 2G_1 \cos \frac{1}{2} \Delta_o$	,,		− ·151
$\theta \pm 2G_1 \sin \frac{1}{2} \Delta_o$	,,		− ·151
λ $F_4 \cos 2 \Delta_o$	,,		− ·003
μ $F_4 \sin 2 \Delta_o$	,,		·000
ν $\frac{1}{2} G_4 \cos 2 \Delta_o$	,,		+ ·024
ρ $\frac{1}{2} G_4 \sin 2 \Delta_o$	,,		·000
$\zeta - \eta$	,,		− ·390
$\frac{1}{2} (\zeta - \eta)$	,,		− ·195
$\beta - \gamma$	,,		+ ·195
$2 (\mu - \nu)$	,,		− ·048
Sum D_o	,,		− ·048
$(\epsilon + \theta)$	,,		− ·692
$\frac{1}{4} (\epsilon + \theta)$	,,		− ·173
$\alpha + \delta$	,,		+ ·627
$4 (\lambda + \rho)$	,,		− ·012
Sum − E_o	,,		+ ·442
E_o	,,		− ·442

Upper sign to left; lower to right.

Evanescent Tide (continued).			
	90° or 6^h.		
$\delta \Delta = \frac{D_o}{E_o}$	,,		+ ·110
(In degrees) $\delta \Delta$	,,		+ 6°·3
$\Delta = \Delta_o + \delta \Delta$	,,		+ 96·3
$\frac{1}{30} \Delta$	,,		3·210
$\frac{1}{30^2} \frac{21}{20} \Delta$	,,		·112
I	,,		3·322
δT	,,		+ ·030
			3·352
Mean int. i	,,		7·900
			11·252
			11^h 15^m·1
(See below) δI	,,		+ 2·5
$\mathfrak{J}$	,,		11^h 18^m
A_2	·691	F_2	− ·195
A_1	+ ·417	F_1	− ·765
A_4	− ·012	F_4	− ·003
Δ	,,		96° or π −84°
$A_2 \cos \Delta$	,,		− ·072
$- F_2 \sin \Delta$	,,		− ·194
$\pm A_1 \cos \frac{1}{2} \Delta$	,,		− ·279
$\mp 2 F_1 \sin \frac{1}{2} \Delta$	,,		+ 1·137
$A_4 \cos 2\Delta$	,,		− ·012
$- \frac{1}{2} F_4 \sin 2\Delta$	,,		+ ·001
Sum h	,,		+ ·581
B_o	,,		3·791
			4·372
(See below) δh	,,		+ ·011
$\mathfrak{H}$	,,		4·38

Upper sign to left; lower to right.

Corrections for Reference to Moon's Transit. (See (57)–(64), § 10.)

Of these corrections δI, δh, M″, N″, P″, Q″ have already been used in the preceding calculation, and we have to show how they are to be computed; we also have to compute $\mathfrak{t}$ and $\mathfrak{h}'$.

From March "intervals" and "heights" we extract I and h, and arrange them in double columns—the even entries in one column and the odd in another. The columns 0^h to 11^h afford the 12 values for 0^h to 11^h of I and h by means of their left hand entries, and they afford the 12 values for 12^h to 23^h by means of their right hand entries. The entry for 23^h is repeated at the top and that for 0^h at the bottom, so that each column has 13 entries, and thus each provides 12 first differences. After finding these differences the distinction of odd and even entries is unnecessary.

The numerical factors 2·2, $\frac{1}{20}$, 2·34, $\frac{11}{300}$, $\frac{1}{20}$, 2·34 in the legends at the top of the columns are absolute constants.

The Θ's are derived from the sequence in Table IV., beginning the sequence with the month treated.

The values of c and d are derived from Table V., for the month named; thus for March c is $-3^m{\cdot}7$ and d is $-0^h{\cdot}062$.

The values of M″, − N″, δI are found in columns vii., xi., xii. of the first table, and P″, − Q″, δh in vii., xi., xii. in the second table. The entries opposite 0^h to 11^h were used above on the left-hand side of columns 0^h to 11^h, and the entries opposite 12^h to 23^h were used above on the right-hand side of the columns 0^h to 12^h.

The quantity $\mathfrak{h}'$ is not a final result, but after *interpolated* values of $\mathfrak{h}'$ (see below on Interpolation) have been found, we shall add to it *computed* values of the annual tide, so as to form $\mathfrak{h}$.

The arithmetical processes involved in these tables are sufficiently explained by the instructions at the head of each column.

March.—Intervals.

	i.	2 dI/dT. ii.	iii.	iv.	v.	vi.	M″. vii.	viii.	ix.	x.	− N″. xi.	δI. xii.	ꞁ. xiii.
	I.	Diff. i.	2·2 × ii.	Θ for Mar.	iii. sin iv.	$\frac{1}{20}$ v.	v. + vi.	iii. cos iv.	$\frac{1}{20}$ viii.	viii. − ix.	iii. + x.	2·34 × v.	c for Mar. × ii.
h.	h. h.	h.		°			m.				m.	m.	m.
23	+ 0·73												
0	+ 0·20	− 1·04	− 2·29	− 10	+ 0·40	+ 2	+ 0·4	− 2·26	− 11	− 2·15	− 4·4	+ 0·9	+ 3·8
1	− 0·31	− 1·02	− 2·24	+ 20	− 0·77	− 4	− 0·8	− 2·11	− 11	− 2·00	− 4·2	− 1·8	+ 3·8
2	− 0·82	− 1·00	− 2·20	+ 50	− 1·69	− 8	− 1·8	− 1·41	− 7	− 1·34	− 3·5	− 4·0	+ 3·7
3	− 1·31	− 0·96	− 2·11	+ 80	− 2·09	− 10	− 2·2	− 0·37	− 2	− 0·35	− 2·5	− 4·9	+ 3·6
&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.
22	+ 1·27	− 1·09	− 2·40	− 70	+ 2·26	+ 11	+ 2·4	− 0·82	− 4	− 0·78	− 3·2	+ 5·3	+ 4·0
23	+ 0·73	− 1·07	− 2·35	− 40	+ 1·51	+ 8	+ 1·6	− 1·80	− 9	− 1·71	− 4·1	+ 3·5	+ 4·0
0	+ 0·20												

March.—Heights.

	i.	2 dh/dT. ii.	iii.	iv.	v.	vi.	P″. vii.	viii.	ix.	x.	− Q″. xi.	δh. xii.	ƅ′. xiii.
	h.	Diff. i.	$\frac{11}{300}$ × ii.	Θ for Mar.	iii. sin iv.	$\frac{1}{20}$ v.	v. + vi.	iii. cos iv.	$\frac{1}{20}$ viii.	viii. − ix.	iii. + x.	2·34 × v.	d for Mar. × ii.
h.	ft. ft.			°			ft.				ft.	ft.	ft.
23	2·53												
0	2·69	+ ·26	+ ·0095	− 10	− ·0017	− 1	− ·002	+ ·0094	+ 5	+ ·0089	+ ·018	− ·004	− ·016
1	2·79	+ ·12	+ 44	+ 20	+ 15	+ 1	+ 2	+ 41	+ 2	+ 39	+ 8	+ 4	− 7
2	2·81	− ·08	− 29	+ 50	− 22	− 1	− 2	− 19	− 1	− 18	− 5	− 5	+ 5
3	2·71	− ·32	− 117	+ 80	− 115	− 6	− 12	− 20	− 1	− 19	− 14	− 27	+ 20
&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.
22	2·28	+ ·56	+ 206	− 70	− 194	− 10	− 20	+ 71	+ 4	+ 67	+ 27	− 45	− 35
23	2·53	+ ·41	+ 150	− 40	− 96	− 5	− 10	+ 115	+ 6	+ 109	+ 26	− 23	− 25
0	2·69												

Additional Values of Intervals and Heights.

Where the intervals change largely between one column and the next, it would add much to the accuracy if additional values were calculated. Thus, in the calculation for March further values between 75° or 5^h and 90° or 6^h, and again, between 90° or 6^h and 105° or 7^h, would be desirable. The like is true for August, where a column between 105° or 7^h and 120° or 8^h would be useful. I choose this last case for my example.

If INMAN'S traverse table be used, interpolation may be made for $112\frac{1}{2}°$ without much difficulty, but I think it is better to interpolate for an even number of additional degrees, and to compute a column for 113°, found by adding 8° to 105°.

It is proposed then to add a new column between the 8th and 9th.

We begin by interpolating in the sequences of angles. In each sequence we have to find the 8th entry for August; then, if it is semi-diurnal, add 16°; if diurnal add 8° for K_1 and P, and subtract 8° for O; and, if quater-diurnal, add 32°. In the sequence for Θ we add 16°, since it is similar to a semi-diurnal term. The calculation runs thus :—

	K_2.	S_2.	O.	K_1.	P.	S_4.
8th entry . .	$\pi - 46°$	$\pi + 19°$	$-88°$	$\pi + 63°$	$\pi - 45°$	$\pi + 83°$
Add	$+16$	$+16$	-8	$+8$	$+8$	$+32$
ϑ	$\pi - 30$	$\pi + 35$	$\pi + 84$	$\pi + 71$	$\pi - 37$	-75

Also $\vartheta_m = 0$, $\vartheta_{2m} = \pi - 36°$, as before.

The 8th entry of Θ is $\pi - 40°$, to which we add 16°, and find $\Theta = \pi - 24°$. With this value of Θ, compute δT, δM, δN. The interpolation amongst the sequences of angles for the parallactic terms is done in the same way. With these new values, and with the former H, Hp^2, Hp we now compute new A's, F's, G's, and are then in a position to compute a new column corresponding to 113° or $7^h\ 32^m$.

In computing δI, δh, M″, N″, P″, Q″, 𝔦, 𝔥′, column ii., for intervals, or $2dI/d$T, must be put equal to $2(I_{120} - I_{105})$; and similarly, column ii. for heights, or $2dh/d$T, must be put equal to $2(h_{120} - h_{105})$.

This interpolation would be especially valuable in the case of M, N, P, Q, which change abruptly.

Some interpolation of the kind has been done in my example, but I do not reproduce the work.

The Calculation of a Low Water Table (referred to in general rule VII.).

This may be done almost independently of the H. W. table by replacing the ϑ's by θ's and using the rules given in § 12. The calculation may, however, be materially abridged, and I will now go over the several steps of the calculation noting the mode of transition from one case to the other.

The new A, G, F will be distinguished from the old by enclosing the new ones in square parentheses.

Semi-diurnal $[A_2]$, $[G_2]$, $[F_2]$.

These sequences are derivable directly from the old ones by the rule

$$[A_2] = A_2 + \tfrac{1}{9}F_2, \qquad [G_2] = G_2 + \tfrac{1}{9}F_2, \qquad [F_2] = F_2 - \tfrac{1}{9}(A_{,,} + A_s).$$

Diurnal $[A_1]$, $[G_1]$, $[F_1]$.

The rule is here more complex :—

$$[A_o] = -\{A_o - \tfrac{1}{8}F_o\}, \quad [G_o] = -\{G_o - \tfrac{1}{35}F_o\}, \quad [F_o] = -\{F_o + \tfrac{1}{35}A_o\},$$
$$[A_,] = A_, + \tfrac{1}{9}F_,, \quad [G_,] = G_, + \tfrac{1}{32}F_,, \quad [F_,] = F_, - \tfrac{1}{32}A_,,$$
$$[A_p] = A_p + \tfrac{1}{10}F_p, \quad [G_p] = G_p + \tfrac{1}{38}F_p, \quad [F_p] = F_p - \tfrac{1}{38}A_p;$$

and in all these sequences *shift the list of months* in the margin *six places downwards,* so that in the O and K_1 sequences March stands where June stood, and in the P sequence March stands where December stood.

If it be agreed to neglect the terms involving $\frac{1}{8}$, $\frac{1}{35}$, &c., the rule is simply to shift the months and change the signs of the O sequences; but at Aden where the diurnal tide is very large, this would lead to a sensible error.

After the new sequences for O, K_1, P have been found, they are combined to find $[A_1]$, $[G_1]$, $[F_1]$ just as for H. W.

Quater-diurnal

$$[A_4] = -A_4, \qquad [G_4] = -G_4, \qquad [F_4] = -F_4;$$

that is to say, simply change signs.

Semi-annual and Mean Water and Annual Tide.

The old calculation serves again.

Mean Interval and its Parallactic Correction.

Here we subtract 180° from κ_m; thus

$$\kappa_m - \pi = 49°$$
$$\tfrac{1}{30}(\kappa_m - \pi) = 1\cdot633$$
$$\tfrac{21}{20}\cdot\tfrac{1}{30^2}(\kappa_m - \pi) = 57$$
$$i = 1\cdot690$$

$$\text{Parallactic correction} = +\,0^{m}\cdot08\,i = +\,0^{m}\cdot14.$$

It may be well to warn the computer that i may be negative, that is to say L. W. may occur on the average earlier than moon's transit.

Parallactic Corrections.

$\alpha, \alpha', \beta, \beta'$ are unchanged.

The rules are

$$[_m\mathrm{A}] = {}_m\mathrm{A} - \tfrac{1}{17}\,{}_m\mathrm{F}, \qquad [_{\prime\prime}\mathrm{A}] = {}_{\prime\prime}\mathrm{A} + \tfrac{2}{9}\,{}_{\prime\prime}\mathrm{F}$$
$$[_m\mathrm{G}] = {}_m\mathrm{G} - \tfrac{1}{17}\,{}_m\mathrm{F}, \qquad [_{\prime\prime}\mathrm{G}] = {}_{\prime\prime}\mathrm{G} + \tfrac{1}{4}\,{}_{\prime\prime}\mathrm{F}$$
$$[_m\mathrm{F}] = {}_m\mathrm{F} + \tfrac{1}{17}\,{}_m\mathrm{A}, \qquad [_{\prime\prime}\mathrm{F}] = {}_{\prime\prime}\mathrm{F} - \tfrac{1}{4}\,{}_{\prime\prime}\mathrm{A}$$

We then compute $[z_2]$, $[m_2]$, $[l_2]$ by the same rules as before.

In the diurnal terms compute by the following rules :—

$$[_o\mathrm{A}] = -\{_o\mathrm{A} - \tfrac{1}{3}\,{}_o\mathrm{F}\}, \qquad [_\prime\mathrm{A}] = {}_\prime\mathrm{A} + \tfrac{2}{9}\,{}_\prime\mathrm{F},$$
$$[_o\mathrm{G}] = -\{_o\mathrm{G} - \tfrac{1}{14}\,{}_o\mathrm{F}\}, \qquad [_\prime\mathrm{G}] = {}_\prime\mathrm{G} + \tfrac{1}{16}\,{}_\prime\mathrm{F},$$
$$[_o\mathrm{F}] = -\{_o\mathrm{F} + \tfrac{1}{14}\,{}_o\mathrm{A}\}, \qquad [_\prime\mathrm{F}] = {}_\prime\mathrm{F} - \tfrac{1}{16}\,{}_\prime\mathrm{A},$$

the list of months in the margin being shifted six places downwards, so that March stands where June stood. The values of $[z_1]$, $[m_1]$, $[l_1]$ are then computed by the same rule as before.

Nodal Corrections.

We may, with sufficient accuracy, take a_2, b_2, c_2, d_2, e_2, f_2 to be unchanged. Referring to the instructions for the computation of a_1, b_1, &c., the new rule may be stated thus :—

$[a_1]$ is (ii.) — (i.), $[b_1]$ is (iv.) — (iii.), $[c_1]$ is (vi.) — (v.),
$[d_1]$ is (viii.)—(vii.), $[e_1]$ is (x.)—(ix.), and $[f_1]$ is (xii.) — (xi.),

and the list of months in the margin is pushed down six places, so that March stands where June stood.

The corrections δT, δM, δN remain unchanged.

When the L. W. sequences have been formed the calculation follows the lines of H. W. calculation precisely, save in three respects—first, in the "heights" h is to be subtracted from B_o, and δh is to be then subtracted from $B_o - h$, instead of the corresponding additions in the H. W. calculation; secondly, the signs of P, Q, S are to be changed as a last step in the calculation of those quantities, in order that the corrections to the heights may be additive instead of subtractive as they would be if we left off exactly as in the case of H. W.; thirdly, after the final table for $\mathfrak{h}'$ has been made, its values must be subtracted from the annual tide.

The reader will easily understand the necessity for these changes when it is remarked that h, δh, $\mathfrak{h}'$ have been estimated as depressions below mean water, whereas B_o and the annual tide are estimated as elevations above the adopted datum; in the result we require, of course, to estimate heights with reference to the datum.

It may be well to warn the computer that $i + I$ may often be negative.

It will be unnecessary to refer henceforth to L. W., since the instructions for H. W. serve also for L. W.

Interpolation.

The sun's longitude increased by 5° is indicated by $\odot$, and the months March, April, May, &c., really mean the dates when $\odot$ is 0°, 30°, 60°, &c.—that is to say, about the middle of the months. The dates which, on the average, fall the nearest to these times, are given in Table VI.

The 12 columns for any month, headed $0^h, 1^h, 2^h, \ldots 11^h$ contain on the left the 12 values of $\mathfrak{I}$, $\mathfrak{H}$, M, N, &c., corresponding to moon's transit at $0^h, 1^h, 2^h, \ldots 11^h$, and they contain on the right the 12 values for moon's transit at $12^h, 13^h, \ldots 23^h$. This applies to the month named at the head of the table. But these values also appertain to the opposite month (*i.e.*, September opposite to March, October to April, and so on) by reading the right hand entries as appertaining to $0^h, 1^h, \ldots 11^h$, and the left to $12^h, 13^h, \ldots 23^h$. The same is true of $\mathfrak{i}$ and $\mathfrak{h}'$ (see *Corrections for reference to moon's transit*), except that here the values change sign in the opposite month; thus the values of $\mathfrak{i}$ and $\mathfrak{h}'$ which we have computed for March must be taken with the opposite sign when applied for September.

Now it is required to form interpolated tables for every 10° of $\odot$, and in all the 18 tables (of which 6 will be originally computed and 12 interpolated) to interpolate for every 20^m of moon's transit.

These interpolations may be done graphically, and I find with millimetre-square paper a convenient scale for $\odot$ is 1 mm. to 1°, and for time of moon's transit 15 mm. to 1^h. These will be set out horizontally as abscissæ, and the ordinates will be time in treating $\mathfrak{I}$, and height in treating $\mathfrak{H}$.

A convenient time scale for $\mathfrak{I}$ is 30 mm. to the hour. In the case of $\mathfrak{H}$ the scale must depend on the range of tide at the place—for Aden (with a small range) I have found 50 mm. to the foot convenient.

I will begin with interpolation for $\odot$ (*i.e.*, for time of year), and will only refer to $\mathfrak{I}$, since $\mathfrak{H}$ follows the same plan. Write March, April, May, ... January, February, March, at 0, 30, ... 360 mm. along a horizontal line, corresponding to the same number of degrees of $\odot$. It may be well to repeat February before the first March, and April after the last March. Set off as an ordinate the left-hand entries from column 0^h for the six months March, April, ... August, and from the right-hand entries of 0^h for the same six months set off ordinates for their opposite months, *e.g.*, the right-hand $\mathfrak{I}$ of 0^h for March, affords $\mathfrak{I}$ of 0^h for September. Through the tops of these ordinates draw a smooth curve of 0^h.* Proceed similarly to form curves of 1^h, 2^h, &c., twelve in all. If the figures get confused we may have two or more, and confusion may often be avoided by drawing parts of the curve with upward or downward shift, so as to make things clear where a number of curves go through nearly the same point.

We now start a fresh figure with time of moon's transit as horizontal line, and $\mathfrak{J}$ as ordinate corresponding to March (or $\odot = 0°$). These 24 (computed) values of $\mathfrak{J}$, joined by a smooth curve, enable us to read off the values of $\mathfrak{J}$ for $\odot = 0°$ for every 20^m of moon's transit, *i.e.*, on the adopted scale, at every 5 mm. of horizontal space.

We now set off from the previous figure the 24 (interpolated) values of $\mathfrak{J}$ corresponding to $\odot = 10°$, of which the first 12 are found at "March + 10 mm." and the last 12 at "September + 10 mm." These 24 values being joined by a curve, give $\mathfrak{J}$ for $\odot = 10°$, and for every 20^m of moon's transit.

We next set off 24 values of $\mathfrak{J}$ corresponding to $\odot = 20°$, of which the first 12 are found in the preceding figure at "March + 20 mm.," and the last 12 at "September + 20 mm." These are treated the same way.

The next in the series are the computed April ($\odot = 30°$) $\mathfrak{J}$'s which are set off like the March ones, joined by a curve and read off to each 20^m of moon's transit.

We then take 24 (interpolated) values of $\mathfrak{J}$ from "April + 10 mm." and "October + 10 mm.," to give $\mathfrak{J}$ for $\odot = 40°$. The 24 from "April + 20 mm." and "October + 20 mm." afford $\mathfrak{J}$ for $\odot = 50°$. The next is the computed May ($\odot = 60°$) series, and we so pass on through six months, the last in the set being derived from "August + 20 mm." and "February + 20 mm." corresponding to $\odot = 170°$.

If one person reads the numbers from the figure, whilst another writes, the tabulation may be done very rapidly.

* The following rule is probably known, but I do not know where it has been stated, except in a note of my own in the 'Messenger of Mathematics.' I have found it very useful in drawing good curves.

Rule for Graphical Interpolation half-way between Computed Ordinates.

Draw the polygon (A) joining the tops of a number of equidistant ordinates, and draw the two polygons (B) joining the tops of alternate ordinates. Then every ordinate has marked on it an intercept or sagitta where a side of polygon (B) cuts it. On the half-way ordinates next on each side of a sagitta, set off one-fourth of the sagitta from the points where the two sides of polygon (A) cut those half-way ordinates; the set-off is to be in the direction in which the sagitta would shoot if it were an arrow. When all the quarter sagittas are thus set off, every half-way ordinate (except the first or last of the series) has two points marked on it.

The interpolated curve passes half-way between the pairs of marked points, except in the case of the first and last half-way ordinate, when it passes through the single marked points.

This rule is correct to fourth differences, except in the case of the first and last ordinate, when it is correct only to third differences. In the cases in the text the computed values are cyclical, and there are therefore no first or last.

By means of proportional compasses set to 4, the quarter sagittas may be set off rapidly, and the bisection of the pairs of marked points may be made by eye.

The same process is applied for tabulation of the 𝔐's. We should, in strictness, do the same by M, N, P, Q, R, S, 𝔦, 𝔥′, but it appears unnecessary to work with so much accuracy.

I have done much of the interpolation by simply writing out the computed values of the quantity to be tabulated in a chess-board table with blanks for the interpolated values. If sixteen squares be considered, a computed value will stand at each corner. Then a great many of the interpolated values may be put in by inspection of the march of the quantity in the two directions. In other parts I make a pencil curve, on millimetre-square paper, of four or five adjacent values, and pass a freehand curve through them to fill in the interpolated values; I rub out the curve when used. It must be remembered that close accuracy in these terms would be mere pedantry.

M, N are computed to the decimal of a minute, and the decimal part may be useful for drawing the pencil interpolation curves, but the result should be tabulated only to the nearest minute. Similarly the third decimal in P and Q may be dropped.

The interpolation of 𝔦 and 𝔥′ follows the same plan, but it must be borne in mind that these functions change sign in opposite months, and this consideration is important when we come to interpolate for $\odot =$ August $+ 10°$, and $+ 20°$.

When there is an evanescent tide (as in the case of March, 18^h) the corrections M, N, R become infinite. As a practical solution this is absurd, and the fact is that there may or may not be a H. W. according to the values of ☊ and Π. Again, in other parts of this and other lunations there may be no H. W., although the tide-table predicts one. In all such cases there is a long period of four or five hours duration of nearly slack water, and it is accordingly almost a matter of indifference whether or not a small H. W. is predicted. It would necessitate very laborious computations to make correct predictions in these cases, and the result would not be worth the labour. I have adopted, therefore, a makeshift, and have replaced H. W. by the height of water and time when the rate of change of water level is a minimum.

It has been proposed above that P, Q, S shall be computed with a conjectural $\Delta = + 90°$, and this is better than the plan which I actually adopted in my experimental table for Aden, of which a sample is given below.* The practical point to consider in the present instructions is the manner of treatment of M, N, R about the time of evanescence. I propose then that the gap in the values shall be bridged by a conjectural curve, and that the values be only given in round numbers. For example, for March we have the following values of M :—$(16^h) + 9^m{\cdot}3$, $(17^h) + 28^m{\cdot}5$ (18^h) blank $(19^h) + 45^m{\cdot}0$ $(20^h) + 17^m{\cdot}7$, &c. By drawing a curve I conjecture $+ 50^m$ for the missing value at 18^h.

A comparison with the corresponding complete curves for February and April helps us in filling the gap.

* Thus if any one seeks to verify my table, he will not get *exactly* my values for P, Q, S in the neighbourhood of 18^h.

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After the complete table for $\mathfrak{h}'$ is formed we proceed to add to it the values computed in the table of the annual tide for every 10° of ☉, and so form a table of $\mathfrak{h}$. For example, the first five values of $\mathfrak{h}'$ are − ·02*, − ·01, − ·01, − ·01*, ·00 (of which those marked * are computed), and to these we add ·390, the computed annual tide for March, and obtain ·37, ·38, ·38, ·38, ·39.

The final results are then arranged in a table.* If the L.W. were also computed I should propose that the L.W. and H.W. should be given alternately. The following is a sample of the table computed only for H.W. at Aden :—

PORT of Aden ; High Water Tide-Table.

Times of moon's transit for—		For March 15th (*i.e.*, from sun's long. 350° to 0°), and for September 17th (*i.e.*, from sun's long. 170° to 180°). The upper signs of $\mathfrak{i}$ and $\mathfrak{h}$ apply to March, the lower to September.									
March.	Sept.	$\mathfrak{I}$.	$\mathfrak{i}$.	$\mathfrak{H}$.	$\mathfrak{h}$.	M.	N.	R.	P.	Q.	S.
h. m.	h. m.	h. m.	m.	ft.	ft.	m.	m.	m.	ft.	ft.	ft.
0 0	12 0	8 5	± 4	6·49	± ·37	0	+ 9	− 1·3	+ ·02	− ·08	+ ·14
20	20	7 55	± 4	·54	± ·38	0	+ 9	− 1·0	+ ·03	− ·08	·14
40	40	46	± 4	·57	± ·38	− 1	+ 9	− 0·6	+ ·04	− ·09	·14
1 0	13 0	7 37	± 4	6·59	± ·38	− 1	+ 9	− 0·2	+ ·05	− ·09	·14
20	20	28	± 4	·60	± ·39	− 2	+ 9	+ 0·1	+ ·06	− ·09	·14
40	40	18	± 4	·60	± ·39	− 2	+ 9	+ 0·4	+ ·07	− ·09	·15
&c.		&c.		&c.		&c.		&c.			
17 0	5 0	8 29	∓ 14	4·21	± ·37	+ 28	− 11	+ 2·3	− ·16	− ·01	·09
20	20	9 52	∓ 15	·21	± ·36	+ 35	− 2	− 7·3	− ·10	− ·03	·10
40	40	11 ·2	∓ 14	·27	± ·35	+ 43	+ 20	− 16	− ·05	− ·05	·10
18 0	6 0	11 17	∓ 10	4·38	± ·35	+ 50	+ 30	− 20	·00	− ·06	·09
20	20	15	∓ 6	·53	± ·34	+ 54	+ 25	− 30	+ ·05	− ·05	·08
40	40	7	∓ 1	·71	± ·33	+ 50	+ 20	− 30	+ ·10	− ·04	·06
&c.		&c.		&c.		&c.		&c.			

If ϖ be the moon's parallax at moon's transit in minutes of arc, and ☊ be the longitude of moon's node, the interval is

$$\mathfrak{I} + \mathfrak{i} + \mathrm{M} \cos ☊ + \mathrm{N} \sin ☊ + (\varpi - 57') \mathrm{R},$$

and the height is

$$\mathfrak{H} + \mathfrak{h} + \mathrm{P} \cos ☊ + \mathrm{Q} \sin ☊ + (\varpi - 57') \mathrm{S}.$$

After the table has been completed the computer should test the correctness of the prediction by computing two or three tides in each month, and comparing the results either with the observations from which the harmonic constants were originally derived, or with other known values of high and low water.

* I have found that it is convenient to cut the constituent tables into strips and paste them together again, so as to save much copying and verification.

Examples of the Use of the Table.

From and after the year 1887, the datum for the tide-tables of the Indian Government for Aden has been 0·37 ft. lower than that used in my table; as I am going to compare my results for 1889 with those of the Indian Government, 0·37 ft. will be added to my heights to make the two comparable.

(A) The moon crossed the meridian at Aden on March 17th, 1889, at $0^h\ 11^m$, Aden M. T. Aden is in $3^h\ 0^m$ E. long., and therefore this is about 21^h, March 16th, G. M. T.; whence from the 'Nautical Almanac' we find ϖ the moon's parallax at Aden transit was 58′·2, and $\varpi - 57' = +1{\cdot}2$. The longitude of moon's node was 108°, and $\cos ☊ = -{\cdot}3$, $\sin ☊ = +1{\cdot}0$.

Then referring to our table and interpolating between $0^h\ 0^m$ and $0^h\ 20^m$, and taking the upper signs of $\mathfrak{i}$ and $\mathfrak{h}$, we find

$$\mathfrak{I} + \mathfrak{i} = 8^h\ 4^m, \quad \mathfrak{H} + \mathfrak{h} = 6{\cdot}89 \text{ ft.}, \quad M = 0, \quad N = +9^m, \quad R = -1^m{\cdot}2,$$
$$P = +{\cdot}03 \text{ ft.}, \quad Q = -{\cdot}08 \text{ ft.}, \quad S = +{\cdot}14 \text{ ft.}$$

Hence

$$M \cos ☊ + N \sin ☊ + R(\varpi - 57) = +8^m,$$

and

$$P \cos ☊ + Q \sin ☊ + S(\varpi - 57) + 0{\cdot}37 = +0{\cdot}45 \text{ ft.}$$

Therefore the interval is $8^h\ 12^m$, and the time of H. W. $8^h\ 12^m + 0^h\ 11^m$ or $8^h\ 23^m$ p.m., March 17th; and the height is 7·34 ft. or 7 ft. 4 in.

The Indian tide-table gives as time $8^h\ 12^m$ p.m., and as height 7 ft. 4 in.

(B) On September 17th, 1889, the moon crossed the meridian at $18^h\ 36^m$, ϖ was 54′·2 or $\varpi - 57' = -2{\cdot}8$, $☊ = 98°$, $\cos ☊ = -{\cdot}14$, $\sin ☊ = +1{\cdot}0$.

Interpolating in our table between $18^h\ 20^m$ and $18^h\ 40^m$, and taking the lower signs of $\mathfrak{i}$ and $\mathfrak{h}$, we find

$$\mathfrak{I} - \mathfrak{i} = 11^h\ 22^m, \quad \mathfrak{H} - \mathfrak{h} = 4{\cdot}34, \quad M = +50^m, \quad N = +20^m, \quad R = -30^m,$$
$$P = +{\cdot}10, \quad Q = -{\cdot}04, \quad S = +{\cdot}06.$$

Hence

$$M \cos ☊ + N \sin ☊ + R(\varpi - 57) = +97^m = +1^h\ 37^m,$$

and

$$P \cos ☊ + Q \sin ☊ + S(\varpi - 57) + 0{\cdot}37 = +0{\cdot}15 \text{ ft.}$$

Therefore the interval is $12^h\ 59^m$, and the time of H. W. $12^h\ 59^m + 18^h\ 36^m$ or $31^h\ 35^m$ September 17th, *i.e.*, $7^h\ 35^m$ p.m. September 18th, and the height is 4·49 or 4 ft. 6 in.

The Indian tide-table gives "no inferior H. W."

This example shows our table at its worst, for it is clear that a nominally small correction to the time which amounts to an hour and forty minutes must give unsatisfactory results. At this time of year mean water has a height of 3 ft. 10 in., hence our prediction only shows a rise of 8 inches. Although there was probably in reality no maximum (as predicted in the Indian table), I should expect that the water stood at about 4 ft. 6 in. at half-past seven of the evening of September 18th, 1889.

Part III.—Comparison and Discussion.

Comparison.

As stated in the Introduction, Mr. Allnutt computed a complete H. W. tide-table for 1889 for Aden in order to compare the results with the Indian tide-tables, which are made with the tide-predicting instrument. When this comparison was made our tables had not been brought into exactly the form given above, and Mr. Allnutt's work was considerably more laborious than it would have been if undertaken later.

The mechanical predictions were probably made with constants which are the means of the results derived from eight years of observation, whereas the constants used in our tables are derived from only four years. Mr. Roberts has supplied me with six weeks of prediction for the year 1887, worked mechanically in duplicate, namely, with the eight-year and the four-year constants. In the latter case the times of H. W. seem to run about 6^m later than in the former, but the difference often rises to 10^m, occasionally to 15^m, and at rare intervals to 20^m. The two sets of constants also give a systematic difference in the heights amounting to about 2 inches, but the difference often rises to 3 inches or falls to 1 inch, and occasionally reaches 4 inches and zero. It follows, therefore, that a sensible part of the discrepancy, or error as it may be called, of our computation is due to the difference of constants. But not nearly all of the "error" can be set down to this cause; it is due to a combination of causes, namely, flaws in the interpolation, imperfect representation of the elliptic tides, and partial inclusion in our method of all the lunar inequalities which are totally neglected in the machine. The principal cause of "error," however, is the imperfect correction for longitude of moon's node and parallax about the time in each lunation when there is partial or total evanescence of the inferior H. W.

Omitting, as we must do, the cases of evanescence, there were 689 H. W. computed by Mr. Allnutt, and he finds "the probable errors" in the time and height to be 9^m and 1·2 inches.

In the course of the year there were thirty-two occasions on which the time "error" amounted to 30^m or more. All of these occur in the inferior H. W. at the times of approximate evanescence, where the nodal and parallactic corrections are large. At these times there is always a period of four or five hours of nearly slack water, and the time at which a small maximum occurs is of no importance from a navigational point of view. If these thirty-two cases be taken away the probable error falls to 7^m A cursory inspection of the table shows also that nearly all the "errors" of 25^m to 30^m fall about the time of evanescence and are therefore unimportant. We are

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accordingly justified in saying that our predictions do not differ materially from the Indian tables.

It has been already mentioned that I have six weeks of mechanical prediction for 1887, made with the identical constants used in our table. I have, therefore, taken a month, or 58 H. W., out of these six weeks, and compared them with my predictions, made without cross interpolation for date. I find that the errors of time are 27 from 0^m to 5^m, 13 from 5^m to 10^m, 10 from 10^m to 15^m, 4 from 15^m to 20^m, one error of 26^m, and one of 34^m. These give a probable error of 7^m. All the large errors fall on the inferior H. W., at the time when it is very small, and they are, therefore, practically unimportant.

In the heights there are 16 cases of agreement, 22 errors of 1 in., 15 of 2 in., and 5 of 3 in. These give a probable error of 1·0 inch.

The errors of 3 inches all fall about the time when the moon's parallax was small, and I also observe that this was commonly a time when the height errors rose to their greatest in 1889. This is probably due to the imperfect representation of the elliptic tides, which, as shown in § 3, is inherent to our method.

The concordance between the two is good enough, but less perfect in the heights than I expected.

The last comparison is between our predictions and actuality. The observed times and heights for part of 1884 have been furnished me by Colonel HILL, R.E., by the direction of Colonel STRAHAN, R.E., Deputy Surveyor-General in India. For the purpose of testing the present method, I have computed a H. W. tide-table from 10th March to 9th April, and again from 12th November to 12th December, 1884. In these periods, there are 117 actual and one evanescent H. W. The observation of one H. W. is missing through an accident to the tide-gauge; there are, therefore, 117 H. W. for the purpose of comparison.*

The following is a table of errors, regardless of signs :—

Time.		Height.	
Magnitude of error.	No. of H. W.	Magnitude of error.	No. of H. W.
m. m.		Inches.	
0 to 5	35	0	15
5 „ 10	32	1	48
10 „ 15	19	2	28
15 „ 20	19	3	14
20 „ 25	5	4	11
26 , 28	2		
33 , 36	2		
56 , 57	2		
Evanescence	1	Evanescence	1
	117		117

* [I originally compared my computations with some results sent me some years ago by Colonel BAIRD, R.E., but as it appeared that these estimates of actuality had probably been biassed in favour of the Indian predictions, I asked to have the tide-curves re-measured.—March 21, 1891.]

Omitting the case of evanescence and assuming the errors to conform in distribution to the normal exponential law (which is not accurately the case), the probable error is about 9^m in time, and 1·4 inch in height.

When the rise from the higher L. W. to the lower H. W. is *nil* there is evanescence, and when that rise is small there is approximate evanescence. I have accordingly examined the 12 cases in which the error of time is equal to or greater than 20^m, and the following table gives the result.

Rise from L. W. to H. W.	Time errors.
Nil.	. . .
6^{in} to 8^{in}	22, 26, 28, 56, 57 minutes
13^{in}	36 minutes
17^{in}	22 ,,
19^{in}	33 ,,
$2^{ft}\ 10^{in}$	22 ,,
$3^{ft}\ 9^{in}$	23 ,,
$3^{ft}\ 11^{in}$	20 ,,

It appears that where the errors of time were 56^m and 57^m the tide was very nearly evanescent, and that the two other considerable errors of time, viz., 36^m and 33^m, pertain to very small tides.

It has been already pointed out that in such cases a considerable error in the time is of no importance, and it is justifiable, in testing the calculations, to set aside the nine tides in which the rise is less or equal to 19 inches.

There remain 108 H. W., and the greatest error in the times amounts to only 23 minutes. In 58 cases the error is 7 minutes or less, and in 51 cases it is 6 minutes or less; as the half of 108 is 54, it follows that the probable error is a little over 6 minutes.

The Indian predictions maintain their standard of excellence fairly well through the periods of approximate evanescence, but out of 116 tides there are 59 cases with time errors of 10 minutes or less; as the half of 116 is 58, the probable error is about 10 minutes.

Turning now to the heights I find that both mine and the Indian predictions present 63 out of 116 H. W. with zero errors or errors of 1 inch. We may take it then that the probable error for both modes of prediction is about 1 inch. The Indian predictions have, however, the disadvantage that several errors of 5, 6, 7 inches occur. On the other hand, the 11 cases of 4 inches of error which occur in mine have a systematic character; they are all positive (actuality the greater), and all but one affect the higher H. W. about the time when the moon's parallax is small. This defect is doubtless due to the imperfect representation of the elliptic, evectional, and variational tides inherent to my method.

The slight superiority shown over the mechanical prediction must be attributed to the fact that I have used better values of the tidal constants than were available in 1883, when the Indian predictions must have been made.

I learn from Colonel HILL that two independent observers reading the same tide-curve will frequently differ by 5^m and sometimes by 10^m in their estimate of the time, and by 1 and sometimes by 2 inches in the height. Accordingly, predictions which agree with a reading of a tide-curve with probable errors of $6\frac{1}{2}^m$ in time and 1 inch in height may claim to possess a high order of accuracy.

I conclude from the preceding discussion that with good values of the tidal constants the present method leads to excellent predictions, and that they are even better than are required for nautical purposes.

Discussion.

It is probable that methods may be invented by which some abridgement of the computations may be made, but I am, of course, unable to suggest such improvements.

The last-mentioned comparison seemed to show that but little accuracy would be lost if P, Q, were entirely omitted, and if $\mathfrak{h}'$ were treated as zero, so that $\mathfrak{h}$ would consist simply of the annual tide. Indeed, the only advantage gained by the retention of P, Q, $\mathfrak{h}'$ appeared to be the avoidance of a few considerable errors in the inferior H. W. about the time of approximate evanescence. Experience must decide whether the computation and the tables may be lightened by the omission of these quantities.

The advantage gained from M and N is marked, but as these quantities arise almost entirely from the diurnal tides, I am inclined to think that, at places where the diurnal tide is not extremely large, a very fair tide-table might be made without them.

The present method will probably be applied to ports of second rate importance, where there are not sufficient data for very accurate determination of the tidal constants. In such cases it will be best to omit the computation of P, Q, $\mathfrak{h}'$, and to postpone that of M, N, and perhaps also of R and $\mathfrak{t}$, until the simple tide-table has been tested as to its adequacy for navigational purposes. At most places the annual tide is so large that $\mathfrak{h}$ cannot be omitted, and it is impossible to dispense with the value of S. But it is possible that it might suffice to attribute to S a constant value,* although this would certainly cause very perceptible error in the heights of the lower H. W. and higher L. W. A tide-table which only gave $\mathfrak{I}$, $\mathfrak{H}$, $\mathfrak{h}$, and a constant S would be fairly short, even if computed for every ten days in the year; and this would be a great gain.

The question of how far to go in each case must depend on a variety of circum-

* I may suppose the elliptic tides unknown, and I should then take $S = \frac{1}{19}H_m + \cdot036H_{,,}$. For Aden this would give $S = \cdot129$, or say $\frac{1}{8}$.

stances. The most important consideration is, I fear, likely to be the amount of money which can be spent on computation and printing; and after this will come the trustworthiness of the tidal constants and the degree of desirability of an accurate tide-table.

My aim has been to reduce the tables to a simple form, and if, as I imagine, the mathematical capacity of an ordinary ship's captain will suffice for the use of the tables, whether in full or abridged, I have attained the principal object in view.